

Maximum estimates for generalized Forchheimer flows in heterogeneous porous media

Emine Celik and Luan Hoang

November 2, 2015

Department of Mathematics and Statistics, Texas Tech University, Box 41042, Lubbock, TX 79409-1042, U.S.A.

Email addresses: `emine.celik@ttu.edu`, `luan.hoang@ttu.edu`

Abstract

This article continues the study in [3] of generalized Forchheimer flows in heterogeneous porous media. Such flows are used to account for deviations from Darcy's law. In heterogeneous media, the derived nonlinear partial differential equation for the pressure can be singular and degenerate in the spatial variables, in addition to being degenerate for large pressure gradient. Here we obtain the estimates for the L^∞ -norms of the pressure and its time derivative in terms of the initial and the time-dependent boundary data. They are established by implementing De Giorgi's iteration in the context of weighted norms with the weights specifically defined by the Forchheimer equation's coefficient functions. With these weights, we prove suitable weighted parabolic Poincaré-Sobolev inequalities and use them to facilitate the iteration. Moreover, local in time L^∞ -bounds are combined with uniform Gronwall-type energy inequalities to obtain long-time L^∞ -estimates.

1 Introduction

Studies of fluid flows in porous media usually use the Darcy equation as a law. However, when the Reynolds number is large, this linear equation is not accurate anymore in describing the fluid dynamics. In that case, Forchheimer equations [8, 9] are commonly used instead. Unlike Darcy's equation, these are nonlinear relations between the velocity and pressure gradient. They are also proposed as models for turbulence in porous media, see e.g. [23]. The reader is referred to [1, 11] and [2, 17, 18, 21] for more information about the Forchheimer flows and their generalizations.

Compared to the Darcy flows, mathematical analysis of the Forchheimer models is scarce. Moreover, previous mathematical works on Forchheimer flows only consider the homogeneous porous media, see e.g. [19, 22] for incompressible fluids, [1, 10–12, 15] for slightly compressible fluids, and [4] for isentropic gases. The problem of Forchheimer flows in heterogeneous media, which is encountered frequently in real life applications, was started in [3]. The current article is a continuation of [3] and is focused on the L^∞ -estimates rather than L^2 . Below, we follow [3] in presenting the model and deriving the key partial differential equation (PDE).

Let a porous medium be modeled as a bounded domain U in space \mathbb{R}^n with C^1 -boundary $\Gamma = \partial U$. Throughout this paper, $n \geq 2$ even though for physics problems $n = 2$ or 3 . Let $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$ be the spatial and time variables. The porosity of this heterogeneous media is denoted by $\phi = \phi(x)$ which depends on the location x .

For a fluid flow in the media, we denote the velocity by $v(x, t) \in \mathbb{R}^n$, pressure by $p(x, t) \in \mathbb{R}$ and density by $\rho(x, t) \in \mathbb{R}^+ = [0, \infty)$.

A generalized Forchheimer equation is

$$g(x, |v|)v = -\nabla p, \quad (1.1)$$

where $g(x, s) \geq 0$ is a function defined on $\bar{U} \times \mathbb{R}^+$. Here, we focus on the case when the function g in (1.1) is of the form

$$g(x, s) = a_0(x)s^{\alpha_0} + a_1(x)s^{\alpha_1} + \cdots + a_N(x)s^{\alpha_N} \quad \text{for } s \geq 0, \quad (1.2)$$

where $N \geq 1, \alpha_0 = 0 < \alpha_1 < \cdots < \alpha_N$ are fixed real numbers, the coefficient functions $a_1(x), a_2(x), \dots, a_{N-1}(x)$ are non-negative, and $a_0(x), a_N(x)$ are positive. The number α_N is the degree of g and is denoted by $\deg(g)$.

Equation (1.1) with g defined by (1.2) is a generalization of Darcy and Forchheimer equations [1, 10, 11]. For instance, when

$$g(x, s) = \alpha, \quad \alpha + \beta s, \quad \alpha + \beta s + \gamma s^2, \quad \alpha + \gamma_m s^{m-1}, \quad (1.3)$$

where $\alpha, \beta, \gamma, m \in (1, 2]$, γ_m are empirical constants, we have Darcy's law, Forchheimer's two term, three term and power laws, respectively, for homogeneous media, see e.g. [2, 17]. The dependence of a_i 's on x indicates the media being heterogeneous. The case when $a_i(x)$'s are independent of x was studied in depth in [10–12, 14, 15].

From (1.1) one can solve for v in terms of ∇p and obtain the equation

$$v = -K(x, |\nabla p|)\nabla p, \quad (1.4)$$

where the function $K : \bar{U} \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is defined by

$$K(x, \xi) = \frac{1}{g(x, s(x, \xi))} \quad \text{for } x \in \bar{U}, \quad \xi \geq 0, \quad (1.5)$$

with $s = s(x, \xi)$ being the unique non-negative solution of $sg(x, s) = \xi$.

We combine (1.4) with the equation of continuity

$$\phi \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0,$$

and the equation of state which, for (isothermal) slightly compressible fluids, is

$$\frac{1}{\rho} \frac{d\rho}{dp} = \varpi, \quad \text{where the constant compressibility } \varpi > 0 \text{ is small.}$$

With small ϖ , by a slight simplification and time scaling, we derive the following initial boundary value problem (IBVP) for the pressure $p(x, t)$:

$$\begin{cases} \phi(x) \frac{\partial p}{\partial t} = \nabla \cdot (K(x, |\nabla p|)\nabla p) & \text{on } U \times (0, \infty), \\ p = \psi & \text{on } \Gamma \times (0, \infty), \\ p(x, 0) = p_0(x) & \text{on } U, \end{cases} \quad (1.6)$$

where $p_0(x)$ and $\psi(x, t)$ are given initial and boundary data. (See [3] for more details.)

Here afterward, the function $g(x, s)$ in (1.2) is fixed, hence so is $K(x, \xi)$.

Although $\phi(x)$ belongs to $(0, 1)$ in applications, we only assume $\phi(x) > 0$ in this paper.

As noted in [3], the PDE in (1.6) is degenerate in ∇p as $|\nabla p| \rightarrow \infty$, and can be both singular and degenerate in x . For such a nonlinear PDE, finer analysis is needed to deal with different types of degeneracy and singularity. To obtain maximum estimates for the solutions, De Giorgi's iteration is used with suitable weighted norms. Thanks to the structure of our equation, these weights are properly defined based on the functions $\phi(x)$ and $a_i(x)$'s. For such weights, the corresponding weighted energy and gradient estimates were already established in [3]. It turns out that we can obtain the maximum estimates for both p and its time derivative under a slightly more stringent condition than the one imposed in [3], see (4.2) compared to (3.1) below. Then the L^∞ -estimates for large time are derived with the use of the uniform Gronwall-type inequalities from [3].

The paper is organized as follows. In section 2, we establish suitable weighted parabolic Poincaré-Sobolev inequalities. In section 3, we review essential results from [3] that will be needed for the current work. Sections 4 and 5 contain estimates of the L^∞ -norm for p and $\partial p / \partial t$. Local in time estimates are established in Propositions 4.1 and 5.1 by De Giorgi's iteration using appropriate weighted norms and the corresponding Poincaré-Sobolev inequalities in section 2. The main results in terms of initial and boundary data are obtained in Theorems 4.2 and 5.2. Particularly, the asymptotic estimates as time goes to infinity are improved to depend only on the asymptotic behavior of the boundary data. This is done by combining the local in time estimates with uniform Gronwall-type inequalities. It is worth mentioning that our results are applicable to all commonly used Forchheimer's laws. Finally, we remark that in case of homogeneous porous media, estimates for p and its time derivative pave the way for obtaining L^∞ -estimates for the gradient, as well as strong continuous dependence and structural stability, see [13–15]. However, it is not known whether such results still hold true for heterogeneous media in the current study.

2 Auxiliaries

First, we recall some elementary inequalities that will be needed. Let $x, y \geq 0$, then

$$(x + y)^p \leq x^p + y^p \quad \text{for all } 1 \geq p > 0, \quad (2.1)$$

$$(x + y)^p \leq 2^{p-1}(x^p + y^p) \quad \text{for all } p \geq 1, \quad (2.2)$$

$$x^\beta \leq x^\alpha + x^\gamma \quad \text{for all } \gamma \geq \beta \geq \alpha \geq 0, \quad (2.3)$$

$$x^\beta \leq 1 + x^\gamma \quad \text{for all } \gamma \geq \beta \geq 0. \quad (2.4)$$

Also,

$$|x - y|^p \geq 2^{-p+1}|x|^p - |y|^p \quad \text{for all } x, y \in \mathbb{R}^n \text{ and } p \geq 1. \quad (2.5)$$

We establish below some weighted parabolic Poincaré-Sobolev inequalities which are suitable to the PDE in (1.6) and are essential to our L^∞ -estimates.

We recall the standard Sobolev-Poincaré's inequality. Let $\dot{W}^{1,q}(U)$ be the space of functions in $W^{1,q}(U)$ with vanishing traces on the boundary. If $1 \leq q < n$ then

$$\|f\|_{L^{q^*}(U)} \leq c \|\nabla f\|_{L^q(U)} \quad \text{for all } f \in \dot{W}^{1,q}(U), \quad (2.6)$$

where $q^* = nq/(n - q)$, the positive constant c depends on q , n and the domain U . For our problem, we need some weighted versions of this.

If $f(x) \geq 0$ is a L^1 -function on U , then define a measure μ_f on U by

$$d\mu_f = f(x)dx.$$

For any $p \in [1, \infty]$ and a measurable set $E \subset U$, we denote by $L_f^p(E)$ and $\|\cdot\|_{L_f^p(E)}$ the L^p space and, respectively, the L^p norm on E corresponding to the measure μ_f .

Similarly, if $f(x, t) \geq 0$ on $U \times \mathbb{R}$ satisfies $f \in L^1(U \times (t_1, t_2))$ for any real numbers $t_1 < t_2$, then define a measure $\bar{\mu}_f$ on $U \times \mathbb{R}$ by

$$d\bar{\mu}_f = f(x, t)dxdt. \quad (2.7)$$

For any $p \in [1, \infty]$ and a bounded, measurable set $E \subset U \times \mathbb{R}$, we denote by $L_f^p(E)$ and $\|\cdot\|_{L_f^p(E)}$ the L^p space and, respectively, the L^p norm on E corresponding to the measure $\bar{\mu}_f$.

Let $\gamma_1(x), \gamma_2(x) > 0$ be two functions on U . Here is the two-weight Poincaré-Sobolev inequality that we need: There is a positive constant c_0 such that

$$\|u\|_{L_{\gamma_1}^r(U)} \leq c_0 \|\nabla u\|_{L_{\gamma_2}^q(U)} \quad (2.8)$$

for all u belonging to a certain class $\dot{X}_{\gamma_1, \gamma_2}^{r, q}(U)$ containing functions which vanish on the boundary Γ .

For some classes of γ_1, γ_2 , and $\dot{X}_{\gamma_1, \gamma_2}^{r, q}(U)$, see e.g. [5, 7, 20]. For instance, [5] characterizes γ_1 and γ_2 so that (2.8) holds for all u such that its extension to zero outside U belongs to $W^{1,1}(\mathbb{R}^n)$. Of course, there are more than one characterization and one class $\dot{X}_{\gamma_1, \gamma_2}^{r, q}(U)$. To avoid considerations of complicated weighted spaces, we will take (2.8) as our starting point. In Example 2.2 below, we give simple examples for a few classes which are applicable to our particular problem.

Assume (2.8) holds for $\gamma_1(x), \gamma_2(x)$ and a space $\dot{X}_{\gamma_1, \gamma_2}^{r, q}(U)$.

For $T > 0$, denote $Q_T = U \times (0, T)$ and

$$\dot{X}_{\gamma_1, \gamma_2}^{r, q}(Q_T) \stackrel{\text{def}}{=} \left\{ u(x, t) : u(\cdot, t) \in \dot{X}_{\gamma_1, \gamma_2}^{r, q}(U) \text{ for almost all } t \in (0, T) \right\}. \quad (2.9)$$

Let c_0 be the positive constant in (2.8).

Throughout, for convenience, we denote $f(t) = f(\cdot, t)$ for any function $f(x, t)$.

Lemma 2.1. *Let r, q be two numbers satisfying*

$$r > 2, \quad r > q \geq 1. \quad (2.10)$$

Set

$$p = 2 + q(1 - 2/r) = q + 2(1 - q/r). \quad (2.11)$$

If $T > 0$ and $u(x, t) \in \dot{X}_{\gamma_1, \gamma_2}^{r, q}(Q_T)$, then

$$\|u\|_{L_{\gamma_1}^p(Q_T)} \leq c_0^{\frac{q}{p}} \left(\operatorname{ess\,sup}_{0 < t < T} \|u(t)\|_{L_{\gamma_1}^2(U)} \right)^{1 - \frac{q}{p}} \|\nabla u\|_{L_{\gamma_2}^q(Q_T)}^{\frac{q}{p}}. \quad (2.12)$$

Consequently,

$$\|u\|_{L_{\gamma_1}^p(Q_T)} \leq c_0^{\frac{q}{p}} \left(\operatorname{ess\,sup}_{0 < t < T} \|u(t)\|_{L_{\gamma_1}^2(U)} + \|\nabla u\|_{L_{\gamma_2}^q(Q_T)} \right). \quad (2.13)$$

Proof. Condition (2.10) and definition (2.11) imply that $q < p$ and $2 < p < r$. Let $\alpha = 1 - \frac{q}{p}$ and $\beta = \frac{q}{p}$. Then $\alpha, \beta \in (0, 1)$,

$$\alpha + \beta = 1 \quad \text{and} \quad \frac{1}{p} = \frac{\alpha}{2} + \frac{\beta}{r}. \quad (2.14)$$

Then by interpolation inequality and (2.8), we have for almost all $t \in (0, T)$ that

$$\begin{aligned} \left(\int_U |u(t)|^p \gamma_1 dx \right)^{\frac{1}{p}} &\leq \left(\int_U |u(t)|^2 \gamma_1 dx \right)^{\frac{\alpha}{2}} \left(\int_U |u(t)|^r \gamma_1 dx \right)^{\frac{\beta}{r}} \\ &\leq c_0^\beta \left(\int_U |u(t)|^2 \gamma_1 dx \right)^{\frac{\alpha}{2}} \left(\int_U |\nabla u(t)|^q \gamma_2 dx \right)^{\frac{\beta}{q}}. \end{aligned}$$

Taking the power p of both side of the previous inequality and integrating it in t from 0 to T , we have

$$\begin{aligned} \int_0^T \int_U |u|^p \gamma_1 dx dt &\leq c_0^{\beta p} \int_0^T \left(\int_U |u(t)|^2 \gamma_1 dx \right)^{\frac{\alpha p}{2}} \left(\int_U |\nabla u(t)|^q \gamma_2 dx \right)^{\frac{\beta p}{q}} dt \\ &\leq c_0^{\beta p} \left(\operatorname{ess\,sup}_{0 < t < T} \int_U |u(t)|^2 \gamma_1 dx \right)^{\frac{\alpha p}{2}} \int_0^T \left(\int_U |\nabla u(t)|^q \gamma_2 dx \right)^{\frac{\beta p}{q}} dt. \end{aligned} \quad (2.15)$$

Since $\beta p/q = 1$, we obtain

$$\|u\|_{L_{\gamma_1}^p(Q_T)}^p \leq c_0^{\beta p} \operatorname{ess\,sup}_{0 < t < T} \|u\|_{L_{\gamma_1}^2(U)}^{\alpha p} \left(\int_0^T \int_U |\nabla u|^q \gamma_2 dx dt \right)^{\frac{\beta p}{q}}. \quad (2.16)$$

Taking power $1/p$ of (2.16) yields (2.12).

In (2.12), we bound

$$\operatorname{ess\,sup}_{0 < t < T} \|u(t)\|_{L_{\gamma_1}^2(U)} \quad \text{and} \quad \|\nabla u\|_{L_{\gamma_2}^q(Q_T)}$$

by their sum, then (2.13) follows. \square

We will refer to the following inequality as Strict Degree Condition (SDC)

$$\deg(g) < \frac{4}{n-2}. \quad (2.17)$$

Note that in the three dimensional cases ($n=3$), (2.17) reads $\deg(g) < 4$, hence it holds for the commonly used Forchheimer models in (1.3).

Example 2.2. We give examples for the weighted elliptic Poincaré-Sobolev inequality (2.8). The parabolic inequalities in Lemma 2.1, hence, follow correspondingly.

(a) Suppose $q \in [1, n)$ and r is a number in the interval $[1, q^*)$.

Let $q_0 \in [1, q)$ such that $r < q_0^* < q^*$. Assume

$$\int_U \gamma_1(x)^{\frac{q_0^*}{q_0^*-r}} dx, \int_U \gamma_2(x)^{-\frac{q_0}{q_0^*-r}} dx < \infty. \quad (2.18)$$

Let $u \in \dot{W}^{1,q_0}(U)$. By Hölder's inequality with powers $\frac{q_0^*}{r}$ and $\frac{q_0^*}{q_0^*-r}$, we have

$$\left(\int_U |u|^r \gamma_1 dx \right)^{\frac{1}{r}} \leq \left(\int_U |u|^{q_0^*} dx \right)^{\frac{1}{q_0^*}} \left(\int_U \gamma_1^{\frac{q_0^*}{q_0^*-r}} dx \right)^{\frac{q_0^*-r}{q_0^* r}}. \quad (2.19)$$

Applying (2.6) to the first Lebesgue norm on the right-hand side gives

$$\left(\int_U |u|^r \gamma_1 dx \right)^{\frac{1}{r}} \leq c \left(\int_U |\nabla u|^{q_0} dx \right)^{\frac{1}{q_0}} \left(\int_U \gamma_1^{\frac{q_0^*}{q_0^*-r}} dx \right)^{\frac{q_0^*-r}{q_0^* r}}. \quad (2.20)$$

where c is the constant in (2.6) with $q = q_0$. Since $q > q_0$, we bound the first integral on the right-hand side by applying Hölder's inequality to functions $|\nabla u|^{q_0} \gamma_2^{\frac{q_0}{q}}$ and $\gamma_2^{-\frac{q_0}{q}}$ with powers q/q_0 and $q/(q - q_0)$. We obtain

$$\left(\int_U |u|^r \gamma_1 dx \right)^{\frac{1}{r}} \leq c_0 \left(\int_U |\nabla u|^q \gamma_2 dx \right)^{\frac{1}{q}}$$

with

$$c_0 = c \left(\int_U \gamma_2(x)^{-\frac{q_0}{q-q_0}} dx \right)^{\frac{q-q_0}{qq_0}} \left(\int_U \gamma_1(x)^{\frac{q_0^*}{q_0^*-r}} dx \right)^{\frac{q_0^*-r}{q_0^*r}} < \infty. \quad (2.21)$$

Therefore, (2.8) holds with c_0 given by (2.21) and

$$\mathring{X}_{\gamma_1, \gamma_2}^{r, q}(U) = \mathring{W}^{1, q_0}(U) \cap \{u : \nabla u \in L_{\gamma_2}^q(U)\}.$$

(b) In [3], we used the case $r = 2$ and $q = 2 - a$. Condition (2.18) becomes

$$\int_U \gamma_1(x)^{\frac{q_0^*}{q_0^*-2}} dx, \int_U \gamma_2(x)^{-\frac{q_0}{2-a-q_0}} dx < \infty,$$

where $1 \leq q_0 < 2 - a$ such that $q_0^* > 2$.

(c) We consider the case when $r > 2$ and $q = 2 - a$. Assume (SDC). One can easily verify that $2 < (2 - a)^*$. Suppose r is a number in the interval $(2, (2 - a)^*)$. Let $q_0 \in (1, 2 - a)$ such that $r < q_0^* < (2 - a)^*$. Assume

$$\int_U \gamma_1(x)^{\frac{q_0^*}{q_0^*-r}} dx, \int_U \gamma_2(x)^{-\frac{q_0}{2-a-q_0}} dx < \infty. \quad (2.22)$$

Then we obtain

$$\|u\|_{L_{\gamma_1}^r(U)} \leq c_0 \|\nabla u\|_{L_{\gamma_2}^{2-a}(U)}$$

with

$$c_0 = c \left(\int_U \gamma_2(x)^{-\frac{q_0}{2-a-q_0}} dx \right)^{\frac{2-a-q_0}{(2-a)q_0}} \left(\int_U \gamma_1(x)^{\frac{q_0^*}{q_0^*-r}} dx \right)^{\frac{q_0^*-r}{q_0^*r}} < \infty.$$

Lemma 2.3. *Let $r, q, \gamma_1(x), \gamma_2(x), c_0$ be the same as in Lemma 2.1. Let m be a number in (q, r) , and define*

$$p = 2 + m \left(1 - \frac{2}{r} \right). \quad (2.23)$$

Then for any $T > 0$, a function $F(x, t) > 0$ on Q_T , and $u(x, t) \in \mathring{X}_{\gamma_1, \gamma_2}^{r, q}(Q_T)$, one has

$$\|u\|_{L_{\gamma_1}^p(Q_T)} \leq c_0^{\frac{m}{p}} \mathcal{Z}^{\frac{m-q}{pq}} \left(\operatorname{ess\,sup}_{0 < t < T} \|u(t)\|_{L_{\gamma_1}^2(U)} + \|\nabla u\|_{L_F^m(Q_T)} \right), \quad (2.24)$$

where

$$\mathcal{Z} = \operatorname{ess\,sup}_{0 < t < T} \left(\int_U \gamma_2(x)^{\frac{m}{m-q}} F(x, t)^{-\frac{q}{m-q}} \chi_{\operatorname{supp} u}(x, t) dx \right).$$

Proof. Denote

$$[[u]] = \operatorname{ess\,sup}_{0 < t < T} \|u(t)\|_{L_{\gamma_1}^2(U)} + \|\nabla u\|_{L_F^m(Q_T)}.$$

Noting that that $q < m$, we apply Hölder's inequality with powers $\frac{m}{q}$ and $\frac{m}{m-q}$ to functions $F(x, t)^{\frac{q}{m}} |\nabla u|^q$ and $\gamma_2(x) F(x, t)^{-\frac{q}{m}} \chi_{\text{supp}u}$, and obtain

$$\begin{aligned} \int_U |\nabla u(t)|^q \gamma_2(x) dx &\leq \left(\int_U F(x, t) |\nabla u(t)|^m dx \right)^{\frac{q}{m}} \left(\int_U \gamma_2(x)^{\frac{m}{m-q}} F(x, t)^{-\frac{q}{m-q}} \chi_{\text{supp}u}(x, t) dx \right)^{\frac{m-q}{m}} \\ &\leq \left(\int_U F(x, t) |\nabla u(t)|^m dx \right)^{\frac{q}{m}} \mathcal{Z}^{\frac{m-q}{m}} \end{aligned}$$

for almost all $t \in (0, T)$. By definition of p , we have $2 < p < r$. Let α and β be defined as in (2.14). Then combining the preceding inequality with (2.15), we have

$$\|u\|_{L_{\gamma_1}^p(Q_T)}^p \leq c_0^{\beta p} [[u]]^{\alpha p} \mathcal{Z}^{\frac{\beta p(m-q)}{mq}} \int_0^T \left(\int_U F(x, t) |\nabla u|^m dx \right)^{\frac{\beta p}{m}} dt.$$

Note that $\beta p/m = 1$, then

$$\|u\|_{L_{\gamma_1}^p(Q_T)}^p \leq c_0^{\beta p} \mathcal{Z}^{\frac{m-q}{q}} [[u]]^{\alpha p} \left(\int_0^T \int_U F(x, t) |\nabla u|^m dx dt \right)^{\frac{\beta p}{m}}.$$

Since $\left(\int_0^T \int_U F(x, t) |\nabla u|^m dx dt \right)^{\frac{1}{m}} \leq [[u]]$, it follows that

$$\|u\|_{L_{\gamma_1}^p(Q_T)}^p \leq c_0^{\beta p} \mathcal{Z}^{\frac{m-q}{q}} [[u]]^{\alpha p} [[u]]^{\beta p} = c_0^m \mathcal{Z}^{\frac{m-q}{q}} [[u]]^p.$$

Taking power $1/p$ both sides of this inequality yields (2.24). \square

The preceding inequalities will be used with specific weights arising from the coefficients of the Forchheimer equations. We review them here.

The following exponent will be used throughout in our calculations

$$a = \frac{\alpha_N}{\alpha_N + 1} \in (0, 1). \quad (2.25)$$

We recall some properties of $K(x, \xi)$. We have from Lemmas III.5 and III.9 in [1] that

$$-aK(x, \xi) \leq \xi \frac{\partial K(x, \xi)}{\partial \xi} \leq 0 \quad \forall \xi \geq 0. \quad (2.26)$$

This implies $K(x, \xi)$ is decreasing in ξ , hence

$$K(x, \xi) \leq K(x, 0) = \frac{1}{g(x, 0)} = \frac{1}{a_0(x)}. \quad (2.27)$$

Define the main weight functions

$$\begin{aligned} M(x) &= \max\{a_j(x) : j = 0, \dots, N\}, \quad m(x) = \min\{a_0(x), a_N(x)\}, \\ W_1(x) &= \frac{a_N(x)^a}{2NM(x)}, \quad W_2(x) = \frac{NM(x)}{m(x)a_N(x)^{1-a}}. \end{aligned} \quad (2.28)$$

Note that

$$W_1(x)a_N(x)^{2-a} = \frac{a_N(x)^2}{2NM(x)} \leq \frac{a_N(x)}{2N} \leq \frac{a_N(x)}{2}. \quad (2.29)$$

From Lemma 1.1 of [3], we have for all $\xi \geq 0$ that

$$\frac{2W_1(x)}{\xi^a + a_N(x)^a} \leq K(x, \xi) \leq \frac{W_2(x)}{\xi^a} \quad (2.30)$$

and, consequently,

$$W_1(x)\xi^{2-a} - \frac{a_N(x)}{2} \leq K(x, \xi)\xi^2 \leq W_2(x)\xi^{2-a}. \quad (2.31)$$

For any number r in $(1, \infty)$, we denote its conjugate exponent by $r' = r/(r-1)$.

We rewrite Lemma 2.3 for our particular problem with specific weights.

Corollary 2.4. *Let function $K(x, \xi)$ and number $a \in (0, 1)$ be defined by (1.5) and (2.25), respectively. Let $\varphi(x)$ be any positive function on U and $W_1(x)$ be defined in (2.28).*

Assume there are $r > 2$ and $c_0 > 0$ such that

$$\|u\|_{L_\varphi^r(U)} \leq c_0 \|\nabla u\|_{L_{W_1}^{2-a}(U)}$$

for any $u(x)$ belonging to a space $\dot{X}_{\varphi, W_1}^{r, 2-a}(U)$.

Then for any $T > 0$, $u(x, t) \in \dot{X}_{\varphi, W_1}^{r, 2-a}(Q_T)$ and function $f(x, t) \geq 0$ on Q_T , one has

$$\begin{aligned} \|u\|_{L_\varphi^{4/r'}(Q_T)} &\leq c_0^{\frac{r'}{2}} \left(\int_U a_N(x) dx + \operatorname{ess\,sup}_{0 < t < T} \int_U W_1(x) f(x, t)^{2-a} \chi_{\operatorname{supp} u}(x, t) dx \right)^{\frac{ar'}{4(2-a)}} \\ &\quad \cdot \left\{ \operatorname{ess\,sup}_{0 < t < T} \|u(t)\|_{L_\varphi^2(U)} + \left(\int_0^T \int_U K(x, f(x, t)) |\nabla u(x, t)|^2 dx dt \right)^{\frac{1}{2}} \right\}. \end{aligned} \quad (2.32)$$

Proof. Denote $\chi = \chi_{\operatorname{supp} u}$ and

$$[[u]] = \operatorname{ess\,sup}_{0 < t < T} \|u(t)\|_{L_\varphi^2(U)} + \left(\int_0^T \int_U K(x, f(x, t)) |\nabla u(x, t)|^2 dx dt \right)^{\frac{1}{2}}. \quad (2.33)$$

Let $m = 2$, $q = 2 - a$, $\gamma_1(x) = \varphi(x)$ and $\gamma_2(x) = W_1(x)$. Then two numbers r and q satisfy (2.10). The number p defined by (2.23) is

$$p = 4 \left(1 - \frac{1}{r} \right) = \frac{4}{r'}.$$

Let $F(x, t) = K(x, f(x, t))$. By Lemma 2.3, we have following particular version of (2.24)

$$\|u\|_{L_\varphi^p(Q_T)} \leq c_0^{\frac{r'}{2}} [[u]] \operatorname{ess\,sup}_{0 < t < T} \left(\int_U W_1(x)^{\frac{2}{a}} K(x, f(x, t))^{-\frac{2-a}{a}} \chi(x, t) dx \right)^{\frac{ar'}{4(2-a)}}. \quad (2.34)$$

For the last integral using (2.30) and (2.2), we have

$$\begin{aligned} W_1(x)^{\frac{2}{a}} K(x, f(x, t))^{-\frac{2-a}{a}} &\leq W_1(x)^{\frac{2}{a}} \left[\frac{f(x, t)^a + a_N(x)^a}{2W_1(x)} \right]^{\frac{2-a}{a}} \\ &\leq W_1(x) a_N(x)^{2-a} + W_1(x) f(x, t)^{2-a}. \end{aligned}$$

By (2.29), we then have

$$\int_U W_1(x)^{\frac{2}{a}} K(x, f(x, t))^{-\frac{2-a}{a}} \chi(x, t) dx \leq \int_U a_N(x) dx + \int_U W_1(x) f(x, t)^{2-a} \chi(x, t) dx.$$

Hence it follows (2.34) that

$$\|u\|_{L^p_\varphi(Q_T)} \leq c_0^{\frac{r'}{2}} [[u]] \operatorname{ess\,sup}_{0 < t < T} \left(\int_U a_N(x) dx + \int_U W_1(x) f(x, t)^{2-a} \chi(x, t) dx \right)^{\frac{ar'}{4(2-a)}}.$$

Thus we obtain (2.32). \square

The following is a generalization of the convergence of fast decay geometry sequences in Lemma 5.6, Chapter II of [16]. It will be used in our version of De Giorgi's iteration.

Lemma 2.5 (cf. [15], Lemma A.2). *Let $\{Y_i\}_{i=0}^\infty$ be a sequence of non-negative numbers satisfying*

$$Y_{i+1} \leq \sum_{k=1}^m A_k B^i Y_i^{1+\mu_k}, \quad i = 0, 1, 2, \dots,$$

where $B > 1$, $A_k > 0$ and $\mu_k > 0$ for $k = 1, 2, \dots, m$. Let $\mu = \min\{\mu_k : 1 \leq k \leq m\}$.

If $Y_0 \leq \min\{(m^{-1} A_k^{-1} B^{-\frac{1}{\mu}})^{1/\mu_k} : 1 \leq k \leq m\}$ then $\lim_{i \rightarrow \infty} Y_i = 0$.

The following simple property will help simplify large time estimates.

Lemma 2.6 (c.f. [3], Lemma A.4). *Let $f(t) \geq 0$ be a C^1 -function on $(0, \infty)$. Assume*

$$\beta = \limsup_{t \rightarrow \infty} [f'(t)]^- < \infty.$$

Then there is $T > 0$ such that for any $t_2 > t_1 > T$,

$$f(t_1) \leq f(t_2) + (t_2 - t_1)(\beta + 1).$$

3 Reviews

In this section we review previous estimates obtained in [3] for a solution $p(x, t)$ of the IBVP (1.6). They will be needed in sections 4 and 5.

Let $\Psi(x, t)$ be an extension $\psi(x, t)$ from $\Gamma \times (0, \infty)$ to $\bar{U} \times [0, \infty)$. Here, all estimates are stated in terms of Ψ , but can certainly be re-written in terms ψ , see e.g. [10].

Define $\bar{p}(x, t) = p(x, t) - \Psi(x, t)$. Throughout the paper, we derive estimates for \bar{p} . The estimates for p are easily obtained by using the triangle inequality $|p| \leq |\bar{p}| + |\Psi|$.

We assume:

(H1) There is $c_1 > 0$ such that if $u(x)$ vanishes on Γ then

$$\|u\|_{L^2_\phi(U)} \leq c_1 \|\nabla u\|_{L^{2-a}_{W_1}(U)}. \quad (3.1)$$

For the validity of (3.1), see Example 2.2(b) with $\gamma_1 = \phi$ and $\gamma_2 = W_1$.

Here afterward, notation $\|\cdot\|_{L^p_f}$ stands for $\|\cdot\|_{L^p_f(U)}$, and p_t means $\frac{\partial p}{\partial t}$.

Let

$$B_1 = \int_U a_N(x) dx, \quad B_* = \max\{B_1, 1\},$$

and for $t \geq 0$,

$$G(t) = B_* + \|\nabla \Psi(t)\|_{L^{2/a_0}}^2 + \|\nabla \Psi(t)\|_{L^{2-a}_{W_1}}^2 + \|\Psi_t(t)\|_{L^{\frac{2-a}{\phi}}_\phi}^2, \quad G_1(t) = \|\nabla \Psi_t(t)\|_{L^{2/a_0}}^2.$$

Let $\mathcal{M}(t)$ be a continuous function on $[0, \infty)$ that satisfies $\mathcal{M}(t)$ is increasing and $\mathcal{M}(t) \geq G(t)$ for all $t \geq 0$. Denote

$$\mathcal{A} = \limsup_{t \rightarrow \infty} G(t) \quad \text{and} \quad \mathcal{B} = \limsup_{t \rightarrow \infty} [G'(t)]^-.$$

Note that $\mathcal{A}, \mathcal{M}(t) \geq 1, B_1$ for all $t \geq 0$.

In the remainder of this section, the symbol C denotes a generic positive constant which may change its values from place to place, depends on number a in (2.25) and the Sobolev constant c_1 in (3.1), but not on individual functions $\phi(x)$, $a_i(x)$'s, the initial and boundary data.

Theorem 3.1 (c.f. [3], Theorem 2.2). *If $t > 0$ then*

$$\int_U \bar{p}^2(x, t) \phi(x) dx \leq \int_U \bar{p}^2(x, 0) \phi(x) dx + C \mathcal{M}(t)^{\frac{2}{2-a}}. \quad (3.2)$$

If $\mathcal{A} < \infty$ then

$$\limsup_{t \rightarrow \infty} \int_U \bar{p}^2(x, t) \phi(x) dx \leq C \mathcal{A}^{\frac{2}{2-a}}. \quad (3.3)$$

If $\mathcal{B} < \infty$ then there is $T > 0$ such that for all $t > T$

$$\int_U \bar{p}^2(x, t) \phi(x) dx \leq C (\mathcal{B}^{\frac{1}{1-a}} + G(t)^{\frac{2}{2-a}}). \quad (3.4)$$

Next, we recall weighted norm estimates for the pressure's derivatives. The differential inequality (3.6) from [3] reads

$$\begin{aligned} \frac{d}{dt} \left(\int_U \bar{p}^2 \phi dx + \int_U H(x, |\nabla p(x, t)|) dx \right) + \int_U \bar{p}_t^2 \phi dx + \frac{1}{4} \int_U H(x, |\nabla p(x, t)|) dx \\ \leq C(G(t) + G_1(t)). \end{aligned} \quad (3.5)$$

Also, we have an inequality of uniform Gronwall-type from [3, Lemma 3.2] for $t \geq 1$,

$$\begin{aligned} \int_U H(x, |\nabla p(x, t)|) dx + \frac{1}{2} \int_{t-\frac{1}{2}}^t \int_U \bar{p}_t^2(x, \tau) \phi(x) dx d\tau \\ \leq C \left(\int_U \bar{p}^2(x, t-1) \phi(x) dx + \int_{t-1}^t (G(\tau) + G_1(\tau)) d\tau \right). \end{aligned} \quad (3.6)$$

Theorem 3.2 (c.f. [3], Corollary 3.5). *For $t > 0$,*

$$\begin{aligned} \int_U W_1(x) |\nabla p(x, t)|^{2-a} dx \leq e^{-\frac{1}{4}t} \int_U H(x, |\nabla p(x, 0)|) dx \\ + C \left(\int_U \bar{p}^2(x, 0) \phi(x) dx + \mathcal{M}^{\frac{2}{2-a}}(t) + \int_0^t e^{-\frac{1}{4}(t-\tau)} G_1(\tau) d\tau \right). \end{aligned} \quad (3.7)$$

For $t \geq 1$,

$$\int_U W_1(x) |\nabla p(x, t)|^{2-a} dx \leq C \left(\int_U \bar{p}^2(x, 0) \phi(x) dx + \mathcal{M}(t)^{\frac{2}{2-a}} + \int_{t-1}^t G_1(\tau) d\tau \right). \quad (3.8)$$

If $\mathcal{A} < \infty$ then

$$\limsup_{t \rightarrow \infty} \int_U W_1(x) |\nabla p(x, t)|^{2-a} dx \leq C \left(\mathcal{A}^{\frac{2}{2-a}} + \limsup_{t \rightarrow \infty} \int_{t-1}^t G_1(\tau) d\tau \right). \quad (3.9)$$

If $\mathcal{B} < \infty$ then there is $T > 1$ such that for all $t > T$,

$$\int_U W_1(x) |\nabla p(x, t)|^{2-a} dx \leq C \left(\mathcal{B}^{\frac{1}{1-a}} + G(t)^{\frac{2}{2-a}} + \int_{t-1}^t G_1(\tau) d\tau \right). \quad (3.10)$$

4 Maximum estimates for the pressure

We derive L^∞ -estimates for the solution $p(x, t)$ of problem (1.6). Let $p(x, t)$ and $\Psi(x, t)$ be the same as in section 3. Let $\bar{p}(x, t) = p(x, t) - \Psi(x, t)$. Then we have

$$\begin{aligned} \phi(x) \frac{\partial \bar{p}}{\partial t} &= \nabla \cdot (K(x, |\nabla p|) \nabla p) - \phi(x) \Psi_t \quad \text{on } U \times (0, \infty), \\ \bar{p} &= 0 \quad \text{on } \Gamma \times (0, \infty). \end{aligned} \quad (4.1)$$

We will make use of the parabolic Poincaré-Sobolev inequality (2.13). Hence, we assume in this section that

(H2) Function $\phi(x)$ belongs to $L^1(U)$, and there are $r > 2$ and $c_2 > 0$ such that

$$\|u\|_{L_\phi^r(U)} \leq c_2 \|\nabla u\|_{L_{W_1}^{2-a}(U)} \quad (4.2)$$

for functions $u(x)$ that vanish on the boundary Γ .

We have the following remarks on (H2):

- (a) If $\phi(x)$ is the physical porosity function in applications, then $\phi(x) \leq 1$, so it belongs to $L^1(U)$.
- (b) According to Example 2.2(c), the number r exists and inequality (4.2) holds under (SDC) and condition (2.22) with $\gamma_1 = \phi$ and $\gamma_2 = W_1$.
- (c) Since $\phi \in L^1(U)$ and $r > 2$, then, by Hölder's inequality, (H2) implies (H1) and (3.1) holds with

$$c_1 = c_2 \left(\int_U \phi(x) dx \right)^{\frac{1}{2} - \frac{1}{r}}. \quad (4.3)$$

Here afterward, we fix r in (H2) and constant c_2 in (4.2). Note that $r' < 2$.

Denote by r_0 the number p defined by (2.11) with $q = 2 - a$, that is,

$$r_0 = 2 + (2 - a) \left(1 - \frac{2}{r} \right) > 2.$$

The following estimates use a fixed parameter r_1 , which is a number in interval $(1, r_0/2)$.

Proposition 4.1. *For any $T_0 \geq 0$, $T > 0$ and $\theta \in (0, 1)$, one has*

$$\begin{aligned} \|\bar{p}(t)\|_{L^\infty(U \times (T_0 + \theta T, T_0 + T))} &\leq \bar{C} \max\{1, c_2\}^{\frac{2-a}{r_0-2}} \left[(\theta T)^{-\frac{1}{2}} + (\theta T)^{-\frac{1}{2-a}} \right]^{\kappa_1} (1 + \omega_{T_0, T})^{\kappa_2} \\ &\quad \cdot \left(\|\bar{p}\|_{L_\phi^2(U \times (T_0, T_0 + T))}^{\nu_1} + \|\bar{p}\|_{L_\phi^2(U \times (T_0, T_0 + T))}^{\nu_2} \right), \end{aligned} \quad (4.4)$$

where constant $\bar{C} > 0$ is independent of c_2 , T_0 , T , and θ ,

$$\kappa_1 = \frac{r_0}{r_0 - 2}, \quad \kappa_2 = \frac{r_0(r_1 - 1)}{2r_0 + (r_0 - 2)r_1(2 - a)}, \quad \nu_1 = \frac{r_0 - 2r_1}{r_0 + (r_0 - 2)r_1}, \quad \nu_2 = \frac{2(r_0 - 2 + a)}{(2 - a)(r_0 - 2)},$$

and

$$\begin{aligned} \omega_{T_0, T} &= T \int_U a_N(x)^{r'_1} \phi(x)^{1-r'_1} dx + T^{r'_1} \int_{T_0}^{T_0+T} \int_U |\Psi_t(x, t)|^{2r'_1} \phi(x) dx dt \\ &\quad + \int_{T_0}^{T_0+T} \int_U (W_1(x) |\nabla \Psi(x, t)|^{2-a} + a_0(x)^{-1} |\nabla \Psi(x, t)|^2)^{r'_1} \phi(x)^{1-r'_1} dx dt. \end{aligned} \quad (4.5)$$

Proof. We use De Giorgi's iteration, see [6, 16]. Without loss of generality we assume $T_0 = 0$ and $\|\bar{p}\|_{L^2_\phi(U \times (0, T))} > 0$. In the following calculations, generic number $\bar{C} > 0$ and specific constants $\bar{C}_1, \bar{C}_2 > 0$ depend on numbers a, r and r_1 , but not on c_2 in (4.2).

Step 1. For each $k \geq 0$, let $\bar{p}^{(k)} = \max\{\bar{p} - k, 0\}$. Note that $\bar{p}^{(k)} = 0$ on Γ .

Let $\chi_k(x, t)$ denote the characteristic of the set $\{(x, t) \in Q_T : \bar{p}^{(k)}(x, t) > 0\}$.

Let $\zeta = \zeta(t) \geq 0$ be a smooth function on \mathbb{R} with $\zeta(t) = 0$ for $t \leq 0$.

Multiplying equation (4.1) by $\bar{p}^{(k)}\zeta$, then integrating over the domain U , and using integration by parts, we get

$$\frac{1}{2} \int_U \frac{\partial |\bar{p}^{(k)}|^2}{\partial t} \zeta \phi dx = - \int_U K(x, |\nabla p|) \nabla p \cdot \nabla \bar{p}^{(k)} \zeta dx - \int_U \bar{p}^{(k)} \zeta \Psi_t \phi dx.$$

For the first integral on the right-hand side, we write

$$\begin{aligned} \nabla p \cdot \nabla \bar{p}^{(k)} &= \nabla \bar{p} \cdot \nabla \bar{p}^{(k)} + \nabla \Psi \cdot \nabla \bar{p}^{(k)} = |\nabla \bar{p}^{(k)}|^2 + \chi_k \nabla \Psi \cdot \nabla \bar{p}^{(k)} \\ &\geq |\nabla \bar{p}^{(k)}|^2 - \frac{1}{2} (|\nabla \bar{p}^{(k)}|^2 + \chi_k |\nabla \Psi|^2) = \frac{1}{2} |\nabla \bar{p}^{(k)}|^2 - \frac{1}{2} \chi_k |\nabla \Psi|^2. \end{aligned}$$

Hence, we gain

$$\begin{aligned} \frac{1}{2} \int_U \frac{\partial |\bar{p}^{(k)}|^2}{\partial t} \zeta \phi dx &\leq -\frac{1}{2} \int_U K(x, |\nabla p|) |\nabla \bar{p}^{(k)}|^2 \zeta dx \\ &\quad + \frac{1}{2} \int_U K(x, |\nabla p|) |\nabla \Psi|^2 \chi_k \zeta dx + \int_U \bar{p}^{(k)} |\Psi_t| \zeta \phi dx. \end{aligned}$$

Let $\varepsilon > 0$. Using (2.27) to bound $K(x, |\nabla p|)$ in the middle integral on the right-hand side, and applying Cauchy's inequality to the last integral yield

$$\begin{aligned} \int_U \frac{\partial |\bar{p}^{(k)}|^2}{\partial t} \zeta \phi dx + \int_U K(x, |\nabla p|) |\nabla \bar{p}^{(k)}|^2 \zeta dx \\ \leq \varepsilon \int_U |\bar{p}^{(k)}|^2 \zeta \phi dx + \varepsilon^{-1} \int_U \chi_k \cdot |\Psi_t|^2 \zeta \phi dx + \int_U \chi_k \cdot a_0(x)^{-1} |\nabla \Psi|^2 \zeta dx. \end{aligned} \quad (4.6)$$

For the second integral on the left-hand side, using relation (2.30) and triangle inequality, we have

$$\begin{aligned} K(x, |\nabla p|) |\nabla \bar{p}^{(k)}|^2 &\geq \frac{2W_1(x)}{|\nabla p|^a + a_N(x)^a} |\nabla \bar{p}^{(k)}|^2 \\ &\geq \frac{2W_1(x) |\nabla \bar{p}^{(k)}|^2}{(|\nabla \bar{p}| + |\nabla \Psi|)^a + a_N(x)^a} = \frac{2W_1(x) |\nabla \bar{p}^{(k)}|^2}{(|\nabla \bar{p}^{(k)}| + |\nabla \Psi|)^a + a_N(x)^a}. \end{aligned}$$

Applying inequality (2.5) to $|\nabla \bar{p}^{(k)}|^2$ in the numerator gives

$$\begin{aligned} K(x, |\nabla p|) |\nabla \bar{p}^{(k)}|^2 &\geq \frac{W_1(x) (|\nabla \bar{p}^{(k)}| + |\nabla \Psi|)^2}{(|\nabla \bar{p}^{(k)}| + |\nabla \Psi|)^a + a_N(x)^a} - \frac{2W_1(x) |\nabla \Psi|^2}{(|\nabla \bar{p}^{(k)}| + |\nabla \Psi|)^a + a_N(x)^a} \\ &\geq \frac{W_1(x) (|\nabla \bar{p}^{(k)}| + |\nabla \Psi|)^2}{(|\nabla \bar{p}^{(k)}| + |\nabla \Psi|)^a + a_N(x)^a} - 2W_1(x) |\nabla \Psi|^{2-a}. \end{aligned}$$

To bound the first term on the right-hand side, we use the following inequality. For $b > 0$ and $\xi \geq 0$, by considering two cases $\xi < b$ and $\xi \geq b$, one can easily prove that

$$\frac{\xi^2}{\xi^a + b^a} \geq \frac{1}{2} (\xi^{2-a} - b^{2-a}).$$

Applying this inequality to $\xi = |\nabla \bar{p}^{(k)}| + |\nabla \Psi|$ and $b = a_N(x)$, we obtain

$$\begin{aligned} K(x, |\nabla p|) |\nabla \bar{p}^{(k)}|^2 &\geq \frac{W_1(x)}{2} \left[(|\nabla \bar{p}^{(k)}| + |\nabla \Psi|)^{2-a} - a_N(x)^{2-a} \right] - 2W_1(x) |\nabla \Psi|^{2-a} \\ &\geq \frac{1}{2} W_1(x) |\nabla \bar{p}^{(k)}|^{2-a} - \frac{1}{2} W_1(x) a_N(x)^{2-a} - 2W_1(x) |\nabla \Psi|^{2-a}. \end{aligned}$$

Using (2.29) to bound $W_1(x) a_N(x)^{2-a}$ we have

$$K(x, |\nabla p|) |\nabla \bar{p}^{(k)}|^2 \geq \frac{1}{2} W_1(x) |\nabla \bar{p}^{(k)}|^{2-a} - \frac{1}{4} a_N(x) - 2W_1(x) |\nabla \Psi|^{2-a}. \quad (4.7)$$

In (4.6), utilizing (4.7) and using the product rule of derivation for the first term on the left-hand side, we have

$$\begin{aligned} \frac{d}{dt} \int_U |\bar{p}^{(k)}|^2 \zeta \phi dx + \frac{1}{2} \int_U W_1(x) |\nabla \bar{p}^{(k)}|^{2-a} \zeta dx &\leq \varepsilon \int_U |\bar{p}^{(k)}|^2 \zeta \phi dx + \int_U |\bar{p}^{(k)}|^2 |\zeta_t| \phi dx \\ &+ \int_U \chi_k \cdot \left[\frac{1}{4} a_N(x) + 2W_1(x) |\nabla \Psi|^{2-a} + \varepsilon^{-1} |\Psi_t|^2 \phi + a_0(x)^{-1} |\nabla \Psi|^2 \right] \zeta dx. \end{aligned}$$

Then integrating in t , using the fact that $\zeta(0) = 0$ and taking supremum on $(0, T)$, we have

$$\begin{aligned} &\sup_{0 < t < T} \int_U |\bar{p}^{(k)}(x, t)|^2 \zeta(t) \phi(x) dx + \frac{1}{2} \int_0^T \int_U W_1(x) |\nabla \bar{p}^{(k)}|^{2-a} \zeta dx dt \\ &\leq 2\varepsilon T \sup_{0 < t < T} \int_U |\bar{p}^{(k)}(x, t)|^2 \zeta(t) \phi(x) dx + 2 \int_0^T \int_U |\bar{p}^{(k)}|^2 |\zeta_t| \phi dx dt \\ &+ 2 \int_0^T \int_U \chi_k \cdot \left[\frac{1}{4} a_N(x) + 2W_1(x) |\nabla \Psi|^{2-a} + \varepsilon^{-1} |\Psi_t|^2 \phi + a_0(x)^{-1} |\nabla \Psi|^2 \right] \zeta dx dt. \end{aligned}$$

Choosing $\varepsilon = \frac{1}{4T}$, and absorbing the first term on the right-hand side into the left yield

$$\begin{aligned} &\sup_{0 < t < T} \int_U |\bar{p}^{(k)}(x, t)|^2 \zeta(t) \phi(x) dx + \int_0^T \int_U W_1(x) |\nabla \bar{p}^{(k)}|^{2-a} \zeta dx dt \\ &\leq 4 \int_0^T \int_U |\bar{p}^{(k)}|^2 |\zeta_t| \phi dx dt + 16 \int_0^T \int_U \chi_k \cdot \mathcal{E}(x, t) \zeta dx dt, \quad (4.8) \end{aligned}$$

where

$$\mathcal{E}(x, t) = a_N(x) + W_1(x) |\nabla \Psi|^{2-a} + T |\Psi_t|^2 \phi + a_0(x)^{-1} |\nabla \Psi|^2. \quad (4.9)$$

Step 2. We will iterate (4.8) with different values of k and different functions ζ . Let $i \geq 0$ be any integer. Denote $t_i = \theta T (1 - \frac{1}{2^i})$. Then $t_0 = 0 < t_1 < t_2 < \dots < \theta T$ and $t_i \rightarrow \theta T$ as $i \rightarrow \infty$.

Let $\zeta(t)$ be a smooth function from \mathbb{R} to $[0, 1]$ such that

$$\zeta_i(t) = \begin{cases} 0 & \text{for } t \leq t_i \\ 1 & \text{for } t \geq t_{i+1} \end{cases} \quad \text{and} \quad 0 \leq \zeta'_i(t) \leq \frac{2}{t_{i+1} - t_i} = \frac{2^{i+2}}{\theta T} \quad \forall t \in \mathbb{R}.$$

Let M_0 be a fixed positive number that will be determined later.

Define $k_i = M_0(1 - 2^{-i})$ and the set $A_{i,j} = \{(x, t) : p(x, t) > k_i, t \in (t_j, T)\}$ for $i, j \geq 0$.

Applying inequality (4.8) to $k = k_{i+1}$ and $\zeta = \zeta_i \leq 1$ gives

$$\begin{aligned} F_i &\stackrel{\text{def}}{=} \sup_{0 < t < T} \int_U |\bar{p}^{(k_{i+1})}(x, t)|^2 \zeta_i(t) \phi(x) dx + \int_0^T \int_U W_1(x) |\nabla \bar{p}^{(k_{i+1})}(x, t)|^{2-a} \zeta_i(t) dx dt \\ &\leq \bar{C} \int_0^T \int_U |\bar{p}^{(k_{i+1})}|^2 \zeta'_i \phi dx dt + \bar{C} \int_0^T \int_U \chi_{k_{i+1}} \mathcal{E} \zeta_i dx dt. \end{aligned}$$

On the right-hand side, using the properties of ζ_i we bound

$$F_i \leq \bar{C} \frac{2^i}{\theta T} \int_{t_i}^T \int_U |\bar{p}^{(k_{i+1})}|^2 \phi dx dt + \bar{C} \int_{t_i}^T \int_U \chi_{k_{i+1}} \mathcal{E} dx dt.$$

Applying Hölder's inequality with powers r_1 and r'_1 to the last double integral yields

$$F_i \leq \bar{C} \frac{2^i}{\theta T} \int_{t_i}^T \int_U |\bar{p}^{(k_{i+1})}|^2 \phi dx dt + \bar{C} \left(\int_{t_i}^T \int_U \chi_{k_{i+1}}^{r_1} \phi dx dt \right)^{\frac{1}{r_1}} \left(\int_{t_i}^T \int_U \mathcal{E}^{r'_1} \phi^{1-r'_1} dx dt \right)^{\frac{1}{r'_1}}.$$

Denote $\omega_T = \omega_{0,T}$ as defined in (4.5). Then

$$\int_{t_i}^T \int_U \mathcal{E}^{r'_1} \phi^{1-r'_1} dx dt \leq \bar{C} \omega_T. \quad (4.10)$$

Then

$$\begin{aligned} F_i &\leq \bar{C} \frac{2^i}{\theta T} \int_{t_i}^T \int_U |\bar{p}^{(k_{i+1})}|^2 \phi dx dt + \bar{C} \omega_T^{\frac{1}{r'_1}} \left(\int_{t_i}^T \int_U \chi_{k_{i+1}}^{r_1} \phi dx dt \right)^{\frac{1}{r_1}} \\ &\leq \bar{C} \frac{2^i}{\theta T} \|\bar{p}^{(k_{i+1})}\|_{L^2_\phi(A_{i+1,i})}^2 + \bar{C} \omega_T^{\frac{1}{r'_1}} \bar{\mu}(A_{i+1,i})^{\frac{1}{r_1}}, \end{aligned}$$

where $\bar{\mu} = \bar{\mu}_\phi$ is the measure defined in (2.7) with $f(x, t) = \phi(x)$. Since $A_{i+1,i} \subset A_{i,i}$ and $\bar{p}^{(k)}$ is decreasing in k , we derive

$$F_i \leq \bar{C} \frac{2^i}{\theta T} \|\bar{p}^{(k_i)}\|_{L^2_\phi(A_{i,i})}^2 + \bar{C} \omega_T^{\frac{1}{r'_1}} \bar{\mu}(A_{i+1,i})^{\frac{1}{r_1}}. \quad (4.11)$$

We estimate the measure $\bar{\mu}(A_{i+1,i})$. Using $A_{i+1,i} \subset A_{i,i}$ again and definition of $\bar{p}^{(k)}$

$$\|\bar{p}^{(k_i)}\|_{L^2_\phi(A_{i,i})}^2 \geq \|\bar{p}^{(k_i)}\|_{L^2_\phi(A_{i+1,i})}^2 \geq (k_{i+1} - k_i)^2 \bar{\mu}(A_{i+1,i}).$$

This implies

$$\bar{\mu}(A_{i+1,i}) \leq (k_{i+1} - k_i)^{-2} \|\bar{p}^{(k_i)}\|_{L^2_\phi(A_{i,i})}^2 = 4^{i+1} M_0^{-2} \|\bar{p}^{(k_i)}\|_{L^2_\phi(A_{i,i})}^2. \quad (4.12)$$

Then (4.11) yields

$$F_i \leq \bar{C} \frac{2^i}{\theta T} \|\bar{p}^{(k_i)}\|_{L^2_\phi(A_{i,i})}^2 + \bar{C} 4^{\frac{i}{r_1}} M_0^{-\frac{2}{r_1}} \omega_T^{\frac{1}{r'_1}} \|\bar{p}^{(k_i)}\|_{L^2_\phi(A_{i,i})}^{\frac{2}{r_1}}. \quad (4.13)$$

Step 3. Applying inequality (2.13) of Lemma 2.1 to $r > 2$, $q = 2 - a$, the weights $\gamma_1(x) = \phi(x)$, $\gamma_2(x) = W_1(x)$, and the function $u(x, t) = \bar{p}^{(k_{i+1})}(x, t) \zeta_i(t)$, we have

$$\begin{aligned} &\|\bar{p}^{(k_{i+1})} \zeta_i\|_{L^{r_0}_\phi(Q_T)} \\ &\leq c_2^{\frac{2-a}{r_0}} \left[\operatorname{ess\,sup}_{0 < t < T} \|\bar{p}^{(k_{i+1})}(t) \zeta_i(t)\|_{L^2_\phi(U)} + \left(\int_0^T \int_U W_1(x) |\nabla(\bar{p}^{(k_{i+1})} \zeta_i)|^{2-a} dx dt \right)^{\frac{1}{2-a}} \right] \\ &\leq c_2^{\frac{2-a}{r_0}} \left[\sup_{0 < t < T} \left(\int_U |\bar{p}^{(k_{i+1})}(x, t)|^2 \zeta_i(t) \phi(x) dx \right)^{\frac{1}{2}} + \left(\int_0^T \int_U W_1(x) |\nabla \bar{p}^{(k_{i+1})}|^{2-a} \zeta_i dx dt \right)^{\frac{1}{2-a}} \right]. \end{aligned}$$

Above, we used the fact that ζ_i is a function of t only, and $0 \leq \zeta_i \leq 1$. Therefore,

$$\|\bar{p}^{(k_{i+1})}\zeta_i\|_{L_\phi^{r_0}(Q_T)} \leq c_2^{\frac{2-a}{r_0}} (F_i^{\frac{1}{2}} + F_i^{\frac{1}{2-a}}). \quad (4.14)$$

Since $\zeta_i = 1$ on $[t_{i+1}, T]$ and $t_i \leq t_{i+1}$, we have from (4.14) that

$$\|\bar{p}^{(k_{i+1})}\|_{L_\phi^{r_0}(A_{i+1,i+1})} \leq \|\bar{p}^{(k_{i+1})}\zeta_i\|_{L_\phi^{r_0}(Q_T)} \leq c_2^{\frac{2-a}{r_0}} (F_i^{\frac{1}{2}} + F_i^{\frac{1}{2-a}}).$$

By Hölder's inequality and by the fact that $A_{i+1,i+1} \subset A_{i+1,i}$ we have

$$\|\bar{p}^{(k_{i+1})}\|_{L_\phi^2(A_{i+1,i+1})} \leq \bar{\mu}(A_{i+1,i+1})^{\frac{1}{2}-\frac{1}{r_0}} \|\bar{p}^{(k_{i+1})}\|_{L_\phi^{r_0}(A_{i+1,i+1})} \leq c_2^{\frac{2-a}{r_0}} \bar{\mu}(A_{i+1,i})^{\frac{1}{2}-\frac{1}{r_0}} (F_i^{\frac{1}{2}} + F_i^{\frac{1}{2-a}}).$$

Combining this with (4.12), (4.13), and using inequality (2.1) yield

$$\begin{aligned} \|\bar{p}^{(k_{i+1})}\|_{L_\phi^2(A_{i+1,i+1})} &\leq \bar{C} c_2^{\frac{2-a}{r_0}} (4^{i+1} M_0^{-2})^{\frac{1}{2}-\frac{1}{r_0}} \|\bar{p}^{(k_i)}\|_{L_\phi^2(A_{i,i})}^{1-\frac{2}{r_0}} \\ &\quad \cdot \left\{ \left(\frac{2^i}{\theta T} \right)^{\frac{1}{2}} \|\bar{p}^{(k_i)}\|_{L_\phi^2(A_{i,i})} + \left(4^{\frac{i}{r_1}} M_0^{-\frac{2}{r_1}} \right)^{\frac{1}{2}} \omega_T^{\frac{1}{2r_1}} \|\bar{p}^{(k_i)}\|_{L_\phi^2(A_{i,i})}^{\frac{1}{r_1}} \right. \\ &\quad \left. + \left(\frac{2^i}{\theta T} \right)^{\frac{1}{2-a}} \|\bar{p}^{(k_i)}\|_{L_\phi^2(A_{i,i})}^{\frac{2}{2-a}} + \left(4^{\frac{i}{r_1}} M_0^{-\frac{2}{r_1}} \right)^{\frac{1}{2-a}} \omega_T^{\frac{1}{r_1(2-a)}} \|\bar{p}^{(k_i)}\|_{L_\phi^2(A_{i,i})}^{\frac{2}{r_1(2-a)}} \right\}. \end{aligned}$$

For $i \geq 0$, define $Y_i = \|\bar{p}^{(k_i)}\|_{L_\phi^2(A_{i,i})} = \|\bar{p}^{(k_i)}\|_{L_\phi^2(U \times (t_i, T))}$. We write the preceding inequality as

$$Y_{i+1} \leq 4^i \cdot \left\{ D_1 Y_i^{2-\frac{2}{r_0}} + D_2 Y_i^{\frac{1}{r_1}+1-\frac{2}{r_0}} + D_3 Y_i^{\frac{2}{2-a}+1-\frac{2}{r_0}} + D_4 Y_i^{\frac{2}{r_1(2-a)}+1-\frac{2}{r_0}} \right\} \quad (4.15)$$

for all $i \geq 0$, where

$$\begin{aligned} D_1 &= \bar{C}_1 c_2^{\frac{2-a}{r_0}} M_0^{-\frac{r_0-2}{r_0}} (\theta T)^{-\frac{1}{2}}, & D_2 &= \bar{C}_1 c_2^{\frac{2-a}{r_0}} M_0^{-\frac{r_0-2}{r_0}-\frac{1}{r_1}} \omega_T^{\frac{1}{2r_1}}, \\ D_3 &= \bar{C}_1 c_2^{\frac{2-a}{r_0}} M_0^{-\frac{r_0-2}{r_0}} (\theta T)^{-\frac{1}{2-a}}, & D_4 &= \bar{C}_1 c_2^{\frac{2-a}{r_0}} M_0^{-\frac{r_0-2}{r_0}-\frac{2}{r_1(2-a)}} \omega_T^{\frac{1}{r_1(2-a)}} \end{aligned}$$

with some $\bar{C}_1 > 0$. Let

$$e_1 = 1 - \frac{2}{r_0}, \quad e_2 = \frac{1}{r_1} - \frac{2}{r_0}, \quad e_3 = \frac{2}{2-a} - \frac{2}{r_0}, \quad e_4 = \frac{2}{r_1(2-a)} - \frac{2}{r_0}.$$

Then $e_1, e_2, e_3, e_4 > 0$ and (4.15) becomes

$$Y_{i+1} \leq 4^i (D_1 Y_i^{1+e_1} + D_2 Y_i^{1+e_2} + D_3 Y_i^{1+e_3} + D_4 Y_i^{1+e_4}).$$

Step 4. We apply Lemma 2.5 with values $m = 4$, $B = 4 > 1$, $A_k = D_k > 0$ and $\mu_k = e_k$, where $k = 1, 2, 3, 4$. If M_0 is chosen sufficiently large such that

$$Y_0 \leq \bar{C}_2 \min\{D_k^{-1/e_k} : k = 1, 2, 3, 4\} \quad (4.16)$$

with an appropriate positive \bar{C}_2 , then $\lim_{i \rightarrow \infty} Y_i = 0$. This gives

$$\int_{\theta T}^T \int_U |\bar{p}^{(M_0)}|^2 \phi(x) dx dt = 0, \quad (4.17)$$

which implies $\bar{p}^{(M_0)}(x, t)\phi(x) = 0$ a.e. in $U \times (\theta T, T)$. Since $\phi(x) > 0$, we have $\bar{p}^{(M_0)}(x, t) = 0$, or equivalently, $\bar{p}(x, t) \leq M_0$ a.e. in $U \times (\theta T, T)$. Repeating the proof with $-p(x, t)$ replacing $p(x, t)$, and $-\Psi(x, t)$ replacing $\Psi(x, t)$, we obtain

$$|\bar{p}(x, t)| \leq M_0 \quad \text{a.e. in } U \times (\theta T, T). \quad (4.18)$$

It remains to find M_0 that satisfies (4.16). Note that $k_0 = 0$ and $Y_0 \leq \|\bar{p}\|_{L_\phi^2(U \times (0, T))}$. It suffices to have

$$\|\bar{p}\|_{L_\phi^2(U \times (0, T))} \leq \bar{C}_2 \min\{D_k^{-1/e_k} : k = 1, 2, 3, 4\}, \quad (4.19)$$

which is equivalent to

$$\begin{aligned} M_0 &\geq \bar{C} c_2^{\frac{2-a}{r_0-2}} [(\theta T)^{-\frac{1}{2}}]^{\frac{r_0}{r_0-2}} \|\bar{p}\|_{L_\phi^2(U \times (0, T))}^{\frac{e_1 r_0}{r_0-2}}, \\ M_0 &\geq \bar{C} c_2^{\frac{(2-a)r_1}{r_0+(r_0-2)r_1}} \omega_T^{\frac{r_0 r_1}{2r_1'(r_0+(r_0-2)r_1)}} \|\bar{p}\|_{L_\phi^2(U \times (0, T))}^{\frac{e_2 r_1 r_0}{r_0+(r_0-2)r_1}}, \\ M_0 &\geq \bar{C} c_2^{\frac{2-a}{r_0-2}} [(\theta T)^{-\frac{1}{2-a}}]^{\frac{r_0}{r_0-2}} \|\bar{p}\|_{L_\phi^2(U \times (0, T))}^{\frac{e_3 r_0}{r_0-2}}, \\ M_0 &\geq \bar{C} c_2^{\frac{(2-a)^2 r_1}{2r_0+(r_0-2)r_1(2-a)}} \omega_T^{\frac{r_0 r_1}{r_1'(2r_0+(r_0-2)r_1(2-a))}} \|\bar{p}\|_{L_\phi^2(U \times (0, T))}^{\frac{e_4 r_1 r_0(2-a)}{2r_0+(r_0-2)r_1(2-a)}}. \end{aligned} \quad (4.20)$$

We compare the lower bounds of M_0 in (4.20).

For $\|\bar{p}\|_{L_\phi^2(U \times (0, T))}$ -terms, we use inequality (2.3), hence need to identify their maximum and minimum powers. Note that

$$e_3 > e_1 > e_2 \quad \text{and} \quad e_3 > e_4 > e_2.$$

Then

$$\frac{e_3 r_0}{r_0 - 2} > \frac{e_1 r_0}{r_0 - 2} > \frac{e_1 r_1 r_0}{r_0 + (r_0 - 2)r_1} > \frac{e_2 r_1 r_0}{r_0 + (r_0 - 2)r_1}, \quad (4.21)$$

$$\frac{e_3 r_0}{r_0 - 2} > \frac{e_4 r_0}{r_0 - 2} > \frac{e_4 r_1 r_0(2-a)}{2r_0 + (r_0 - 2)r_1(2-a)}. \quad (4.22)$$

Therefore, the maximum power of $\|\bar{p}\|_{L_\phi^2(U \times (0, T))}$ in (4.20) is

$$\frac{e_3 r_0}{r_0 - 2} = \nu_2.$$

To find the minimum power, we compare the smallest powers in (4.21) and (4.22). With explicit calculations

$$\frac{e_4 r_1 r_0(2-a)}{2r_0 + (r_0 - 2)r_1(2-a)} = \frac{r_0 - (2-a)r_1}{r_0 + (r_0 - 2)r_1(2-a)/2} > \frac{r_0 - 2r_1}{r_0 + (r_0 - 2)r_1} = \frac{e_2 r_1 r_0}{r_0 + (r_0 - 2)r_1}.$$

Therefore, the minimum power of $\|\bar{p}\|_{L_\phi^2(U \times (0, T))}$ in (4.20) is

$$\frac{e_2 r_1 r_0}{r_0 + (r_0 - 2)r_1} = \nu_1.$$

For two ω_T -terms in (4.20), the powers of ω_T satisfy

$$\frac{r_0 r_1}{2r_1'(r_0 + (r_0 - 2)r_1)} \leq \frac{r_0 r_1}{r_1'(2r_0 + (r_0 - 2)r_1(2-a))} = \kappa_2.$$

For (θT) -terms, we just add $(\theta T)^{-\frac{1}{2-a}}$ to $(\theta T)^{-\frac{1}{2}}$. Also, the maximum power of c_2 is $\frac{2-a}{r_0-2}$. In summary, each lower bound in (4.20) is less than or equal to

$$\bar{C} \max\{1, c_2\}^{\frac{2-a}{r_0-2}} (1 + \omega_T)^{\kappa_2} \left\{ (\theta T)^{-\frac{1}{2}} + (\theta T)^{-\frac{1}{2-a}} \right\}^{\frac{r_0}{r_0-2}} \left\{ \|\bar{p}\|_{L_\phi^2(U \times (0, T))}^{\nu_1} + \|\bar{p}\|_{L_\phi^2(U \times (0, T))}^{\nu_2} \right\}.$$

Hence we choose this value as M_0 , and obtain (4.4) from (4.18). The proof is complete. \square

We combine Proposition 4.1 with estimates in section 3 to give L^∞ -estimates in terms of the initial and boundary data. Define for $t > s \geq 0$,

$$\begin{aligned} N_1(s, t) = & \max \left\{ 1, \int_U a_N(x)^{r'_1} \phi(x)^{1-r'_1} dx \right\} \\ & + \int_s^t \int_U \left[\left(W_1(x) |\nabla \Psi(x, \tau)|^{2-a} + a_0(x)^{-1} |\nabla \Psi(x, \tau)|^2 \right)^{r'_1} \phi(x)^{1-r'_1} + |\Psi_t(x, \tau)|^{2r'_1} \phi(x) \right] dx d\tau. \end{aligned} \quad (4.23)$$

Then $N_1(s, t) \geq 1$ and $\omega_{T_0, T}$ in (4.5) satisfies

$$\omega_{T_0, T} \leq 2 \max\{1, T\}^{r'_1} N_1(T_0, T_0 + T). \quad (4.24)$$

In the next theorem, we assume also (H1). The generic positive constant C depends on a, r, r_1, c_1 in (3.1), and c_2 in (4.2), but not on individual functions $\phi(x)$, $a_i(x)$'s, the initial and boundary data.

Theorem 4.2. *Let the notation be the same as in Proposition 4.1 and Theorem 3.1.*

(i) *If $t \in (0, 1)$, then*

$$\|\bar{p}\|_{L^\infty(U \times (t/2, t))} \leq C t^{-\kappa_3} N_1(0, t)^{\kappa_2} \left(\|\bar{p}(0)\|_{L_\phi^2(U)} + \mathcal{M}(t)^{\frac{1}{2-a}} \right)^{\nu_2}, \quad (4.25)$$

where

$$\kappa_3 = \frac{\kappa_1}{2-a} - \frac{\nu_1}{2} = \frac{r_0}{(2-a)(r_0-2)} - \frac{r_0-2r_1}{2(r_0+(r_0-2)r_1)} > 0. \quad (4.26)$$

If $t \geq 1$, then

$$\|\bar{p}\|_{L^\infty(U \times (t-\frac{1}{2}, t))} \leq C N_1(t-1, t)^{\kappa_2} \left(\|\bar{p}(0)\|_{L_\phi^2(U)} + \mathcal{M}(t)^{\frac{1}{2-a}} \right)^{\nu_2}. \quad (4.27)$$

(ii) *If $\mathcal{A} < \infty$ then*

$$\limsup_{t \rightarrow \infty} \|\bar{p}\|_{L^\infty(U \times (t-\frac{1}{2}, t))} \leq C \left(\limsup_{t \rightarrow \infty} N_1(t-1, t) \right)^{\kappa_2} \mathcal{A}^{\frac{\nu_2}{2-a}}. \quad (4.28)$$

(iii) *If $\mathcal{B} < \infty$ then there is $T > 0$ such that for all $t > T$*

$$\|\bar{p}\|_{L^\infty(U \times (t-\frac{1}{2}, t))} \leq C N_1(t-1, t)^{\kappa_2} \left(\mathcal{B}^{\frac{1}{2(1-a)}} + G(t)^{\frac{1}{2-a}} \right)^{\nu_2}. \quad (4.29)$$

Proof. By remark (c) after (H2), the condition (H1) in section 3 is met with constant c_1 now specified by (4.3). Thus, all constants C 's in estimates of section 3 now depend on this c_1 .

(i) Let $t \in (0, 1)$. Applying inequality (4.4) to $T_0 = 0$, $T = t < 1$ and $\theta = 1/2$, and taking into account estimate (4.24), we have

$$\|\bar{p}\|_{L^\infty(U \times (t/2, t))} \leq C(t^{-\frac{1}{2}} + t^{-\frac{1}{2-a}})^{\kappa_1} N_1(0, t)^{\kappa_2} \left(t^{\nu_1/2} \sup_{0 \leq \tau \leq t} \|\bar{p}(\tau)\|_{L_\phi^2(U)}^{\nu_1} + t^{\nu_2/2} \sup_{0 \leq \tau \leq t} \|\bar{p}(\tau)\|_{L_\phi^2(U)}^{\nu_2} \right).$$

Noticing from Proposition (4.1) that $\nu_2 \geq \nu_1$, we apply inequality (2.4) to $x = \|\bar{p}\|_{L_\phi^2(U)}$, $\beta = \nu_1$ and $\gamma = \nu_2$, and derive from the preceding inequality that

$$\|\bar{p}\|_{L^\infty(U \times (t/2, t))} \leq C t^{-\frac{\kappa_1}{2-a} + \frac{\nu_1}{2}} N_1(0, t)^{\kappa_2} \left(1 + \sup_{0 \leq \tau \leq t} \|\bar{p}(\tau)\|_{L_\phi^2(U)}^{\nu_2} \right). \quad (4.30)$$

Using estimate (3.2) for $\|\bar{p}\|_{L_\phi^2(U)}$, and the fact $\mathcal{M}(t) \geq 1$, we obtain (4.25) from (4.30).

Next, let $t \in [1, \infty)$. Applying inequality (4.4) to $T_0 = t - 1$, $T = 1$ and $\theta = 1/2$, and using (4.24) again, we have

$$\begin{aligned} \|\bar{p}\|_{L^\infty(U \times (t-\frac{1}{2}, t))} &\leq C N_1(t-1, t)^{\kappa_2} \left(\sup_{\tau \in [t-1, t]} \|\bar{p}(\tau)\|_{L_\phi^2(U)}^{\nu_1} + \sup_{\tau \in [t-1, t]} \|\bar{p}(\tau)\|_{L_\phi^2(U)}^{\nu_2} \right) \\ &\leq C N_1(t-1, t)^{\kappa_2} \left(1 + \sup_{\tau \in [t-1, t]} \|\bar{p}(\tau)\|_{L_\phi^2(U)}^{\nu_2} \right). \end{aligned} \quad (4.31)$$

Again, using (3.2) to estimate $\|\bar{p}\|_{L_\phi^2}$ in (4.31), we obtain (4.27).

(ii) Taking the limit superior of (4.31) as $t \rightarrow \infty$, we have

$$\limsup_{t \rightarrow \infty} \|\bar{p}\|_{L^\infty(U \times (t-\frac{1}{2}, t))} \leq C \limsup_{t \rightarrow \infty} N_1(t-1, t)^{\kappa_2} \left(1 + \limsup_{t \rightarrow \infty} \sup_{\tau \in [t-1, t]} \|\bar{p}(\tau)\|_{L_\phi^2(U)}^{\nu_2} \right).$$

By the limit estimate (3.3) and the fact $\mathcal{A} \geq 1$, we obtain (4.28).

(iii) Using estimate (3.4) in (4.31), we have for sufficiently large t that

$$\|\bar{p}\|_{L^\infty(U \times (t-\frac{1}{2}, t))} \leq C N_1(t-1, t)^{\kappa_2} \left(1 + \mathcal{B}^{\frac{1}{1-a}} + \sup_{\tau \in [t-1, t]} G(\tau)^{\frac{2}{2-a}} \right)^{\nu_2/2}. \quad (4.32)$$

By Lemma 2.6, one has for $\tau \in [t-1, t]$ that

$$G(\tau) \leq G(t) + (t - \tau)(\mathcal{B} + 1) \leq G(t) + \mathcal{B} + 1.$$

Using this to estimate the sum on the right-hand side of (4.32) gives

$$\begin{aligned} 1 + \mathcal{B}^{\frac{1}{1-a}} + \sup_{\tau \in [t-1, t]} G(\tau)^{\frac{2}{2-a}} &\leq 1 + \mathcal{B}^{\frac{1}{1-a}} + (G(t) + \mathcal{B} + 1)^{\frac{2}{2-a}} \\ &\leq C(1 + \mathcal{B}^{\frac{1}{1-a}} + \mathcal{B}^{\frac{2}{2-a}} + G(t)^{\frac{2}{2-a}}) \leq C(\mathcal{B}^{\frac{1}{1-a}} + G(t)^{\frac{2}{2-a}}). \end{aligned} \quad (4.33)$$

The last inequality uses (2.4) with $x = \mathcal{B}$, $\beta = 2/(2-a)$ and $\gamma = 1/(1-a)$, combined with the fact $G(t) \geq 1$. Thus, desired estimate (4.29) follows (4.32) and (4.33). \square

5 Maximum estimates for the pressure's time derivative

Let $p(x, t)$, $\Psi(x, t)$ and $\bar{p}(x, t)$ be as in section 4. Define

$$q(x, t) = p_t(x, t) \quad \text{and} \quad \bar{q}(x, t) = \bar{p}_t(x, t) = p_t(x, t) - \Psi_t.$$

We will estimate L^∞ -norm of \bar{q} .

Assume (H2) again with fixed number $r > 2$ and Sobolev constant c_2 in (4.2).

In the following, we also fix a number r_2 such that

$$r_2 > \frac{2}{2 - r'} = \frac{2(r - 1)}{r - 2}.$$

Note that its conjugate exponent r'_2 belongs to $(1, 2/r')$.

First, we establish a counter part of Proposition 4.1.

Proposition 5.1. *There is a constant $\bar{C} > 0$ independent of c_2 such that for any $T_0 \geq 0$, $T > 0$ and $\theta \in (0, 1)$ we have*

$$\begin{aligned} \|\bar{p}_t\|_{L^\infty(U \times (T_0 + \theta T, T_0 + T))} &\leq \bar{C} \max\{1, c_2\}^{\frac{r}{r-2}} \left([(\theta T)^{-\frac{1}{2}} \mathcal{S}_{T_0, T, \theta}]^{\frac{1}{\delta_1}} + (Z_{T_0, T} \mathcal{S}_{T_0, T, \theta})^{\frac{1}{1+\delta_2}} \right) \\ &\quad \cdot \left(\|\bar{p}_t\|_{L_\phi^2(U \times (T_0, T_0 + T))} + \|\bar{p}_t\|_{L_\phi^2(U \times (T_0, T_0 + T))}^{\frac{\delta_2}{1+\delta_2}} \right), \end{aligned} \quad (5.1)$$

where

$$\begin{aligned} \delta_1 &= 1 - \frac{r'}{2}, \quad \delta_2 = \frac{1}{r'_2} - \frac{r'}{2}, \\ \mathcal{S}_{T_0, T, \theta} &= \left(B_1 + \sup_{t \in [T_0 + \theta T, T_0 + T]} \int_U W_1(x) |\nabla p(x, t)|^{2-a} dx \right)^{\frac{ar'}{4(2-a)}}, \\ Z_{T_0, T} &= \|a_0(x)^{-1/2} \nabla \Psi_t\|_{L_\phi^{2r_2}(U \times (T_0, T_0 + T))} + T^{1/2} \|\Psi_{tt}\|_{L_\phi^{2r_2}(U \times (T_0, T_0 + T))}. \end{aligned}$$

Proof. Again, we assume, without loss of generality, that $T_0 = 0$ and $\|\bar{p}_t\|_{L_\phi^2(U \times (0, T))} > 0$. The function $\bar{q}(x, t)$ solves

$$\begin{aligned} \phi(x) \frac{\partial \bar{q}}{\partial t} &= \nabla \cdot (K(x, |\nabla p|) \nabla p)_t - \phi(x) \Psi_{tt} \quad \text{on } U \times (0, \infty), \\ \bar{q} &= 0 \quad \text{on } \Gamma \times (0, \infty). \end{aligned} \quad (5.2)$$

We prove (5.1) by using De Giorgi's iteration for equation (5.2). Below, $\bar{C} > 0$ is generic, while \bar{C}_3, \bar{C}_4 have particular values, and they all depend on a, r, r_2 , but not on c_2 in (4.2).

Step 1. Let $k \geq 0$ be arbitrary, define $\bar{q}^{(k)} = \max\{\bar{q} - k, 0\}$. Denote by $\chi_k(x, t)$ the characteristic function of the set $\{(x, t) \in U \times (0, T) : \bar{q}(x, t) > k\}$.

Let $\zeta = \zeta(t) \geq 0$ be a C^∞ -function on \mathbb{R} with $\zeta(t) = 0$ on $(-\infty, 0]$.

Multiplying (5.2) by the function $\bar{q}^{(k)} \zeta^2$ and integrating over U , we have

$$\int_U \frac{\partial \bar{q}}{\partial t} \bar{q}^{(k)} \zeta^2 \phi dx = \int_U \nabla \cdot (K(x, |\nabla p|) \nabla p)_t \bar{q}^{(k)} \zeta^2 dx - \int_U \Psi_{tt} \bar{q}^{(k)} \zeta^2 \phi dx. \quad (5.3)$$

The integrand on the left-hand side of (5.3) is

$$\frac{1}{2} \frac{\partial |\bar{q}^{(k)}|^2}{\partial t} \cdot \zeta^2 = \frac{1}{2} \frac{\partial}{\partial t} (\bar{q}^{(k)} \zeta)^2 - |\bar{q}^{(k)}|^2 \zeta' \zeta.$$

On the right-hand side of (5.3), we perform integration by parts for the first term and using the fact $\bar{q} = 0$ on the boundary. We obtain

$$\frac{1}{2} \frac{d}{dt} \int_U (\bar{q}^{(k)} \zeta)^2 \phi dx - \int_U |\bar{q}^{(k)}|^2 \zeta' \zeta \phi dx = - \int_U (K(x, |\nabla p|) \nabla p)_t \nabla (\bar{q}^{(k)} \zeta^2) dx - \int_U \Psi_{tt} \bar{q}^{(k)} \zeta^2 \phi dx.$$

Since ζ depends on t only, $\nabla(\bar{q}^{(k)} \zeta^2) = \nabla \bar{q}^{(k)} \zeta^2$. Applying the product rule to the t -derivative in the first integral on the right-hand side gives

$$\frac{1}{2} \frac{d}{dt} \int_U (\bar{q}^{(k)} \zeta)^2 \phi dx = \int_U |\bar{q}^{(k)}|^2 \zeta' \zeta \phi dx - I_1 + I_2 - \int_U \Psi_{tt} \bar{q}^{(k)} \zeta^2 \phi dx, \quad (5.4)$$

where

$$I_1 = \int_U K(x, |\nabla p|) \nabla q \cdot \nabla \bar{q}^{(k)} \zeta^2 dx, \quad I_2 = - \int_U (K(x, |\nabla p|))_t \nabla p \cdot \nabla \bar{q}^{(k)} \zeta^2 dx. \quad (5.5)$$

For I_1 ,

$$\nabla q \cdot \nabla \bar{q}^{(k)} = \nabla \bar{q} \cdot \nabla \bar{q}^{(k)} + \nabla \Psi_t \cdot \nabla \bar{q}^{(k)} = |\nabla \bar{q}^{(k)}|^2 + \nabla \Psi_t \cdot \nabla \bar{q}^{(k)},$$

Hence,

$$-I_1 \leq - \int_U K(x, |\nabla p|) |\nabla \bar{q}^{(k)} \zeta|^2 dx + \int_U K(x, |\nabla p|) |\nabla \Psi_t| |\nabla \bar{q}^{(k)}| \zeta^2 dx. \quad (5.6)$$

For I_2 , by using (2.26):

$$|(K(x, |\nabla p|))_t \nabla p \cdot \nabla \bar{q}^{(k)}| \leq |(K'(x, |\nabla p|))| \frac{|\nabla p \cdot \nabla q| |\nabla p \cdot \nabla \bar{q}^{(k)}|}{|\nabla p|} \leq a K(x, |\nabla p|) |\nabla q| |\nabla \bar{q}^{(k)}|.$$

For the last product,

$$|\nabla q| |\nabla \bar{q}^{(k)}| \leq |\nabla \bar{q}| |\nabla \bar{q}^{(k)}| + |\nabla \Psi_t| |\nabla \bar{q}^{(k)}| = |\nabla \bar{q}^{(k)}|^2 + |\nabla \Psi_t| |\nabla \bar{q}^{(k)}|.$$

Hence,

$$|I_2| \leq a \int_U K(x, |\nabla p|) |\nabla \bar{q}^{(k)} \zeta|^2 dx + a \int_U K(x, |\nabla p|) |\nabla \Psi_t| |\nabla \bar{q}^{(k)}| \zeta^2 dx. \quad (5.7)$$

Then combining (5.4), (5.6) and (5.7) gives

$$-I_1 + I_2 \leq -(1-a) \int_U K(x, |\nabla p|) |\nabla \bar{q}^{(k)} \zeta|^2 dx + (1+a) \int_U K(x, |\nabla p|) |\nabla \Psi_t| |\nabla \bar{q}^{(k)}| \zeta^2 dx. \quad (5.8)$$

For the last integral, applying Cauchy's inequality gives

$$\begin{aligned} (1+a) K(x, |\nabla p|) |\nabla \Psi_t| |\nabla \bar{q}^{(k)}| &\leq \frac{1-a}{2} K(x, |\nabla p|) |\nabla \bar{q}^{(k)}|^2 + \frac{(1+a)^2}{2(1-a)} K(x, |\nabla p|) |\nabla \Psi_t|^2 \chi_k \\ (\text{by using (2.27)}) &\leq \frac{1-a}{2} K(x, |\nabla p|) |\nabla \bar{q}^{(k)}|^2 + \bar{C} a_0(x)^{-1} |\nabla \Psi_t|^2 \chi_k. \end{aligned}$$

Therefore, it follows this, (5.8) and (5.4) that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_U (\bar{q}^{(k)} \zeta)^2 \phi dx + \frac{1-a}{2} \int_U K(x, |\nabla p|) |\nabla (\bar{q}^{(k)} \zeta)|^2 dx \\ \leq \int_U |\bar{q}^{(k)}|^2 \zeta |\zeta'| \phi dx + \bar{C} \int_U a_0(x)^{-1} |\nabla \Psi_t|^2 \chi_k \zeta^2 + \int_U |\Psi_{tt}| \bar{q}^{(k)} \zeta^2 \phi dx. \end{aligned} \quad (5.9)$$

Let $\varepsilon > 0$. Applying Cauchy's inequality to the last integral, we have

$$\int_U |\Psi_{tt}| |\bar{q}^{(k)}| \zeta^2 \phi dx \leq \varepsilon \int_U |\bar{q}^{(k)}| \zeta^2 \phi dx + \frac{1}{4\varepsilon} \int_U |\Psi_{tt}|^2 \chi_k \cdot \zeta^2 \phi dx.$$

Using this estimate in (5.9) we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_U (\bar{q}^{(k)} \zeta)^2 \phi dx + \frac{1-a}{2} \int_U K(x, |\nabla p|) |\nabla(\bar{q}^{(k)} \zeta)|^2 dx \\ & \leq \int_U |\bar{q}^{(k)}|^2 \zeta |\zeta'| \phi dx + \varepsilon \int_U |\bar{q}^{(k)}| \zeta^2 \phi dx + \bar{C} \int_U \left(a_0(x)^{-1} |\nabla \Psi_t|^2 + \varepsilon^{-1} |\Psi_{tt}|^2 \right) \chi_k \cdot \zeta^2 \phi dx. \end{aligned} \quad (5.10)$$

Integrating (5.10) in time from 0 to t and then taking supremum on $(0, T)$, we find

$$\begin{aligned} & \sup_{0 < t < T} \int_U |\bar{q}^{(k)}(x, t) \zeta(t)|^2 \phi(x) dx + (1-a) \int_0^T \int_U K(x, |\nabla p|) |\nabla(\bar{q}^{(k)} \zeta)|^2 dx dt \\ & \leq 4 \int_0^T \int_U |\bar{q}^{(k)}|^2 \zeta |\zeta'| \phi dx dt + 4\varepsilon T \sup_{0 < t < T} \int_U |\bar{q}^{(k)}(x, t) \zeta(t)|^2 \phi(x) dx \\ & \quad + \bar{C} \int_0^T \int_U (a_0(x)^{-1} |\nabla \Psi_t|^2 + \varepsilon^{-1} |\Psi_{tt}|^2) \chi_k \cdot \zeta^2 \phi dx dt. \end{aligned}$$

By selecting $\varepsilon = \frac{1}{8T}$, it follows that

$$\begin{aligned} & \sup_{0 < t < T} \int_U |\bar{q}^{(k)}(x, t) \zeta(t)|^2 \phi(x) dx + \int_0^T \int_U K(x, |\nabla p|) |\nabla(\bar{q}^{(k)} \zeta)|^2 dx dt \\ & \leq \bar{C} \int_0^T \int_U |\bar{q}^{(k)}|^2 \zeta |\zeta'| \phi dx dt + \bar{C} \int_0^T \int_U E(x, t) \chi_k \cdot \zeta^2 \phi dx dt, \end{aligned} \quad (5.11)$$

where

$$E(x, t) = a_0(x)^{-1} |\nabla \Psi_t|^2 + T |\Psi_{tt}|^2. \quad (5.12)$$

Applying inequality (2.32) of Corollary 2.4 to $u = \bar{q}^{(k)} \zeta$ and $f(x, t) = |\nabla p(x, t)|$, we have

$$\begin{aligned} & \|\bar{q}^{(k)} \zeta\|_{L_{\phi}^{4/r'}(U \times (0, T))} \\ & \leq 2c_2^{\frac{r'}{2}} \mathcal{S} \left(\sup_{0 < t < T} \int_U |\bar{q}^{(k)}(x, t) \zeta(t)|^2 \phi(x) dx + \int_0^T \int_U K(x, |\nabla p(x, t)|) |\nabla \bar{q}^{(k)} \zeta|^2 dx dt \right)^{\frac{1}{2}}, \end{aligned} \quad (5.13)$$

where

$$\mathcal{S} = \left(\int_U a_N(x) dx + \sup_{t \in \text{supp } \zeta \cap (0, T)} \int_U W_1(x) |\nabla p(x, t)|^{2-a} dx \right)^{\frac{ar'}{4(2-a)}}. \quad (5.14)$$

Denote $c_3 = c_2^{\frac{r'}{2}}$. Then combining (5.13) with (5.11) yields

$$\|\bar{q}^{(k)} \zeta\|_{L_{\phi}^{4/r'}(U \times (0, T))} \leq \bar{C} c_3 \mathcal{S} \left(\int_0^T \int_U |\bar{q}^{(k)}|^2 \zeta |\zeta'| \phi dx dt + \int_0^T \int_U E \chi_k \cdot \zeta^2 \phi dx dt \right)^{\frac{1}{2}}. \quad (5.15)$$

Step 2. Let $M_0 > 0$ be a fixed value that will be determined later. For $i \geq 0$, define

$$k_i = M_0 \left(2 - \frac{1}{2^i} \right) \quad \text{and} \quad t_i = \theta T \left(1 - \frac{1}{2^{i+1}} \right). \quad (5.16)$$

Then the sequences $(k_i)_{i \geq 0}$ and $(t_i)_{i \geq 0}$ are strictly increasing with $M_0 \leq k_i < 2M_0$, $\frac{\theta T}{2} \leq t_i < \theta T$, and $\lim_{i \rightarrow \infty} k_i = 2M_0$, $\lim_{i \rightarrow \infty} t_i = \theta T$.

Denote

$$A_{i,j} = \{(x, t) : \bar{q}(x, t) > k_i, t \in (t_j, T)\} \quad \text{for } i, j = 0, 1, 2, 3, \dots$$

Let $\zeta_i(t)$ be a C^∞ -function on \mathbb{R} valued in $[0, 1]$ with the following properties

$$\zeta_i(t) = \begin{cases} 0 & \text{in } t \leq t_i \\ 1 & \text{for } t \geq t_{i+1}, \end{cases} \quad \text{and} \quad 0 \leq \zeta'_i(t) \leq \frac{2}{t_{i+1} - t_i} = \frac{2^{i+3}}{\theta T}. \quad (5.17)$$

Define

$$F_i = \|\bar{q}^{(k_{i+1})} \zeta_i\|_{L_\phi^{4/r'}(A_{i+1,i})} = \|\bar{q}^{(k_{i+1})} \zeta_i\|_{L_\phi^{4/r'}(U \times (0, T))}.$$

Now, we apply inequality (5.15) to $k = k_{i+1}$ and $\zeta = \zeta_i$. Denote $\mathcal{S}_T = \mathcal{S}_{0,T,\theta}$. Then for all $i \geq 0$, the \mathcal{S} defined in (5.14) satisfies $\mathcal{S} \leq \mathcal{S}_T$. Therefore, we have from (5.15) that

$$F_i \leq \bar{C} c_3 \mathcal{S}_T \left(\int_0^T \int_U |\bar{q}^{(k_{i+1})}|^2 \zeta_i \zeta'_i \phi dx dt + \int_0^T \int_U E \chi_{k_{i+1}} \zeta_i^2 \phi dx dt \right)^{\frac{1}{2}}.$$

Using properties of ζ_i in (5.17), we have

$$\begin{aligned} F_i &\leq \bar{C} c_3 \mathcal{S}_T \left(\frac{2^i}{\theta T} \int_0^T \int_U |\bar{q}^{(k_{i+1})}|^2 \zeta_i \phi dx dt + \iint_{A_{i+1,i}} E \phi dx dt \right)^{\frac{1}{2}} \\ &\leq \bar{C} c_3 \mathcal{S}_T \left\{ \frac{2^{\frac{i}{2}}}{(\theta T)^{1/2}} \|\bar{q}^{(k_{i+1})}\|_{L_\phi^2(A_{i+1,i})} + \left(\iint_{A_{i+1,i}} E \phi dx dt \right)^{\frac{1}{2}} \right\}. \end{aligned}$$

For the last integral, applying Hölder's inequality with powers r_2 and r'_2 , we derive

$$F_i \leq \bar{C} c_3 \mathcal{S}_T \left\{ \frac{2^{\frac{i}{2}}}{(\theta T)^{1/2}} \|\bar{q}^{(k_{i+1})}\|_{L_\phi^2(A_{i+1,i})} + \left(\iint_{A_{i+1,i}} E^{r_2} \phi dx dt \right)^{\frac{1}{2r_2}} \left(\iint_{A_{i+1,i}} \phi dx dt \right)^{\frac{1}{2r'_2}} \right\}. \quad (5.18)$$

Let $Z_T = Z_{0,T}$ and recall that the measure $\bar{\mu} = \bar{\mu}_\phi$ is defined in (2.7) with $f(x, t) = \phi(x)$. Clearly,

$$\left(\iint_{A_{i+1,i}} E^{r_2} \phi dx dt \right)^{\frac{1}{2r_2}} \leq \bar{C} Z_T \quad \text{and} \quad \|\bar{q}^{(k_{i+1})}\|_{L_\phi^2(A_{i+1,i})} \leq \|\bar{q}^{(k_i)}\|_{L_\phi^2(A_{i,i})}.$$

Hence, we derive from (5.18) that

$$F_i \leq \bar{C} c_3 \mathcal{S}_T \left\{ 2^i (\theta T)^{-\frac{1}{2}} \|\bar{q}^{(k_i)}\|_{L_\phi^2(A_{i,i})} + Z_T \bar{\mu}(A_{i+1,i})^{\frac{1}{2r'_2}} \right\}. \quad (5.19)$$

Next, by Hölder's inequality and, again, the fact $A_{i+1,i+1} \subset A_{i+1,i}$ one has

$$\begin{aligned} \|\bar{q}^{(k_{i+1})}\|_{L_\phi^2(A_{i+1,i+1})} &\leq \|\bar{q}^{(k_{i+1})}\|_{L_\phi^{4/r'}(A_{i+1,i+1})} \bar{\mu}(A_{i+1,i+1})^{\frac{1}{2} - \frac{r'}{4}} \\ &= \|\bar{q}^{(k_{i+1})} \zeta_i\|_{L_\phi^{4/r'}(A_{i+1,i+1})} \bar{\mu}(A_{i+1,i+1})^{\frac{1}{2} - \frac{r'}{4}} \leq F_i \bar{\mu}(A_{i+1,i})^{\frac{1}{2} - \frac{r'}{4}}. \end{aligned} \quad (5.20)$$

Then (5.19) and (5.20) imply

$$\|\bar{q}^{(k_{i+1})}\|_{L^2_\phi(A_{i+1,i+1})} \leq \bar{C}c_3\mathcal{S}_T\left(2^i(\theta T)^{-\frac{1}{2}}\|\bar{q}^{(k_i)}\|_{L^2_\phi(A_{i,i})}\bar{\mu}(A_{i+1,i})^{\frac{1}{2}-\frac{r'}{4}} + Z_T\bar{\mu}(A_{i+1,i})^{\frac{1}{2}-\frac{r'}{4}+\frac{1}{2r'_2}}\right). \quad (5.21)$$

To estimate the measure $\bar{\mu}(A_{i+1,i})$, note that

$$\|\bar{q}^{(k_i)}\|_{L^2_\phi(A_{i,i})} \geq \|\bar{q}^{(k_i)}\|_{L^2_\phi(A_{i+1,i})} \geq (k_{i+1} - k_i)\bar{\mu}(A_{i+1,i})^{\frac{1}{2}}.$$

Hence

$$\bar{\mu}(A_{i+1,i}) \leq (k_{i+1} - k_i)^{-2}\|\bar{q}^{(k_i)}\|_{L^2_\phi(A_{i,i})}^2 = \frac{4^{i+1}}{M_0^2}\|\bar{q}^{(k_i)}\|_{L^2_\phi(A_{i,i})}^2. \quad (5.22)$$

Now substituting (5.22) into (5.21), we obtain

$$\begin{aligned} \|\bar{q}^{(k_{i+1})}\|_{L^2_\phi(A_{i+1,i+1})} &\leq \bar{C}c_34^i\mathcal{S}_T\left\{(\theta T)^{-\frac{1}{2}}M_0^{-1+\frac{r'}{2}}\|\bar{q}^{(k_i)}\|_{L^2_\phi(A_{i,i})}^{2-\frac{r'}{2}} \right. \\ &\quad \left. + Z_TM_0^{-1+\frac{r'}{2}-\frac{1}{r'_2}}\|\bar{q}^{(k_i)}\|_{L^2_\phi(A_{i,i})}^{1-\frac{r'}{2}+\frac{1}{r'_2}}\right\}. \end{aligned} \quad (5.23)$$

Step 3. Defining $Y_i = \|\bar{q}^{(k_i)}\|_{L^2_\phi(A_{i,i})} = \|\bar{q}^{(k_i)}\|_{L^2_\phi(U \times (t_i, T))}$ for $i \geq 0$. From (5.23), we have for all $i \geq 0$ that

$$Y_{i+1} \leq 4^i(D_1Y_i^{1+\delta_1} + D_2Y_i^{1+\delta_2}), \quad (5.24)$$

where $D_1 = \bar{C}_3c_3\mathcal{S}_T(\theta T)^{-\frac{1}{2}}M_0^{-\delta_1}$ and $D_2 = \bar{C}_3c_3\mathcal{S}_TZ_TM_0^{-1-\delta_2}$, for some $\bar{C}_3 > 0$.

Applying Lemma 2.5 to the sequence $\{Y_i\}_{i=0}^\infty$ and (5.24) with M_0 chosen sufficiently large such that

$$Y_0 \leq \bar{C}_4 \min\{D_1^{-\frac{1}{\delta_1}}, D_2^{-\frac{1}{\delta_2}}\} \quad (5.25)$$

for a particular $\bar{C}_4 > 0$, then $\lim_{i \rightarrow \infty} Y_i = 0$, i.e.,

$$\lim_{i \rightarrow \infty} \|\bar{q}^{(k_i)}\|_{L^2_\phi(A_{i,i})} = \int_{\theta T}^T \int_U |\bar{q}^{(2M_0)}|^2 \phi dx dt = 0. \quad (5.26)$$

Using the same arguments that yield (4.18) from (4.17), here we have from (5.26) that

$$|\bar{q}(x, t)| \leq 2M_0 \quad \text{a.e. in } U \times (\theta T, T). \quad (5.27)$$

Lastly, we find $M_0 > 0$ to satisfy (5.25). Note that $Y_0 \leq \|\bar{q}\|_{L^2_\phi(U \times (0, T))}$. Then a sufficient condition for (5.25) is

$$\|\bar{q}\|_{L^2_\phi(U \times (0, T))} \leq \bar{C}D_1^{-\frac{1}{\delta_1}} \quad \text{and} \quad \|\bar{q}\|_{L^2_\phi(U \times (0, T))} \leq \bar{C}D_2^{-\frac{1}{\delta_2}}.$$

Solving these inequalities gives

$$M_0 \geq \bar{C}(c_3\mathcal{S}_T(\theta T)^{-1/2})^{\frac{1}{\delta_1}}\|\bar{q}\|_{L^2_\phi(U \times (0, T))} \quad \text{and} \quad M_0 \geq \bar{C}(c_3\mathcal{S}_TZ_T)^{\frac{1}{1+\delta_2}}\|\bar{q}\|_{L^2_\phi(U \times (0, T))}^{\frac{\delta_2}{1+\delta_2}}.$$

Since $1 + \delta_2 > \delta_1$, we estimate the c_3 -terms by

$$c_3^{\frac{1}{\delta_1}} + c_3^{\frac{1}{1+\delta_2}} \leq 2 \max\{1, c_3\}^{\frac{1}{\delta_1}} = 2 \max\{1, c_2\}^{\frac{r'}{2-r'}} = 2 \max\{1, c_2\}^{\frac{r}{r-2}}.$$

Hence we choose M_0 as

$$M_0 = \bar{C} \max\{1, c_2\}^{\frac{r}{r-2}} \left((\mathcal{S}_T(\theta T)^{-1/2})^{\frac{1}{\delta_1}} + (\mathcal{S}_T Z_T)^{\frac{1}{1+\delta_2}} \right) \left(\|\bar{q}\|_{L_\phi^2(U \times (0, T))} + \|\bar{q}\|_{L_\phi^2(U \times (0, T))}^{\frac{\delta_2}{1+\delta_2}} \right).$$

Then inequality (5.1) follows (5.27). The proof is complete. \square

Then combining Proposition 5.1 with estimates in section 3 yields specific bounds for \bar{p}_t . For $t > s \geq 0$, define

$$N_2(s, t) = 1 + \|a_0^{-1/2} \nabla \Psi_t\|_{L_\phi^{2r_2}(U \times (s, t))} + \|\Psi_{tt}\|_{L_\phi^{2r_2}(U \times (s, t))}. \quad (5.28)$$

Then $N_2(s, t) \geq 1$ and

$$1 + Z_{T_0, T} \leq (\max\{1, T\})^{1/2} N_2(T_0, T_0 + T). \quad (5.29)$$

Below, we assume (H1) and denote by C a generic positive constant depending on a, r, r_2, c_1 in (3.1), and c_2 in (4.2).

Theorem 5.2. *Let δ_1, δ_2 be as in Proposition 5.1.*

(i) *If $t \in (0, \frac{3}{2})$ then*

$$\|\bar{p}_t\|_{L^\infty(U \times (t/2, t))} \leq C t^{-\frac{1}{2\delta_1}} N_2(0, t)^{\frac{1}{1+\delta_2}} \left(A_0 + \mathcal{M}(t)^{\frac{2}{2-a}} + \int_0^t G_1(\tau) d\tau \right)^{\kappa_4}, \quad (5.30)$$

where

$$\kappa_4 = \frac{1}{2} + \frac{ar}{2(2-a)(r-2)},$$

$$A_0 = \int_U H(x, |\nabla p(x, 0)|) dx + \int_U \bar{p}^2(x, 0) \phi dx.$$

If $t \geq \frac{3}{2}$ then

$$\|\bar{p}_t\|_{L^\infty(U \times (t-\frac{1}{4}, t))} \leq C N_2(t - \frac{1}{2}, t)^{\frac{1}{1+\delta_2}} \left(\|\bar{p}(0)\|_{L_\phi^2}^2 + \mathcal{M}(t)^{\frac{2}{2-a}} + \int_{t-\frac{5}{4}}^t G_1(\tau) d\tau \right)^{\kappa_4}. \quad (5.31)$$

(ii) *If $\mathcal{A} < \infty$ then*

$$\limsup_{t \rightarrow \infty} \|\bar{p}_t\|_{L^\infty(U \times (t-\frac{1}{4}, t))} \leq C \left(\limsup_{t \rightarrow \infty} N_2(t - \frac{1}{2}, t) \right)^{\frac{1}{1+\delta_2}} \left(\mathcal{A}^{\frac{2}{2-a}} + \limsup_{t \rightarrow \infty} \int_{t-1}^t G_1(\tau) d\tau \right)^{\kappa_4}. \quad (5.32)$$

(iii) *If $\mathcal{B} < \infty$ then there is $T > 0$ such that for all $t > T$,*

$$\|\bar{p}_t\|_{L^\infty(U \times (t-\frac{1}{4}, t))} \leq C N_2(t - \frac{1}{2}, t)^{\frac{1}{1+\delta_2}} \left(\mathcal{B}^{\frac{1}{1-a}} + G(t)^{\frac{2}{2-a}} + \int_{t-\frac{5}{4}}^t G_1(\tau) d\tau \right)^{\kappa_4}. \quad (5.33)$$

Proof. Similar to the proof of Theorem 4.2, all constants C 's in section 3 are made dependent of c_1 defined by (4.3).

(i) Let $t \in (0, 3/2]$. We apply (5.1) to $t \in (0, 3/2]$, $T_0 = 0$, $T = t$ and $\theta = 1/2$. On the one hand, the quantity $\mathcal{S}_{T_0, T, \theta} = \mathcal{S}_{0, t, 1/2}$ in (5.1) is bounded by using (3.7):

$$\mathcal{S}_{0, t, 1/2} = \left(B_1 + \sup_{\tau \in [t/2, t]} \int_U W_1(x) |\nabla p(x, \tau)|^{2-a} dx \right)^{\frac{ar'}{4(2-a)}} \leq C S_1(t)^{\frac{ar'}{4(2-a)}},$$

where

$$S_1(t) = A_0 + \mathcal{M}(t)^{\frac{2}{2-a}} + \int_0^t G_1(\tau) d\tau \geq 1.$$

Above, we used the facts $\mathcal{M}(t)$ is increasing and $\mathcal{M}(t) \geq 1, B_1$. On the other hand, to estimate $Z_{T_0, T} = Z_{0, t}$ in (5.1), we use (5.29). These estimates result in

$$\begin{aligned} \|\bar{q}\|_{L^\infty(U \times (t/2, t))} &\leq C \left(t^{-\frac{1}{2\delta_1}} S_1(t)^{\frac{1}{\delta_1} \frac{ar'}{4(2-a)}} + N_2(0, t)^{\frac{1}{1+\delta_2}} S_1(t)^{\frac{1}{1+\delta_2} \frac{ar'}{4(2-a)}} \right) \\ &\quad \cdot \left(\|\bar{q}\|_{L_\phi^2(U \times (0, t))} + \|\bar{q}\|_{L_\phi^2(U \times (0, t))}^{\frac{\delta_2}{1+\delta_2}} \right). \end{aligned} \quad (5.34)$$

Note that $1 + \delta_2 = \delta_1 + 1/r'_2 > \delta_1$. Hence, the maximum power of $S_1(t)$ is

$$\kappa_5 = \frac{ar'}{4(2-a)} \cdot \frac{1}{\delta_1} = \kappa_4 - \frac{1}{2}.$$

For the power $\|\bar{q}\|_{L_\phi^2(U \times (0, t))}$ in (5.34), note that $\frac{\delta_2}{1+\delta_2} \leq 1$. Then it follows

$$\|\bar{q}\|_{L^\infty(U \times (t/2, t))} \leq C t^{-\frac{1}{2\delta_1}} N_2(0, t)^{\frac{1}{1+\delta_2}} S_1(t)^{\kappa_5} \left(1 + \|\bar{q}\|_{L_\phi^2(U \times (0, t))} \right).$$

To estimate $\|\bar{q}\|_{L_\phi^2(U \times (0, t))}$, we integrate (3.5) in time from 0 to t , and have

$$\int_0^t \int_U \bar{q}^2(x, \tau) \phi(x) dx d\tau \leq A_0 + C \int_0^t (G(\tau) + G_1(\tau)) d\tau \leq C S_1(t).$$

Then

$$\|\bar{q}\|_{L^\infty(U \times (t/2, t))} \leq C t^{-\frac{1}{2\delta_1}} N_2(0, t)^{\frac{1}{1+\delta_2}} S_1(t)^{\kappa_5} (1 + S_1(t))^{1/2} \leq C t^{-\frac{1}{2\delta_1}} N_2(0, t)^{\frac{1}{1+\delta_2}} S_1(t)^{\kappa_5 + 1/2},$$

and inequality (5.30) follows.

Now, let $t \geq \frac{3}{2}$. Using (5.1) with $T_0 = t - \frac{1}{2}$, $T = \frac{1}{2}$ and $\theta = 1/2$, and utilizing (5.29), we have

$$\begin{aligned} \|\bar{q}\|_{L^\infty(U \times (t-\frac{1}{4}, t))} &\leq C \left(S_2(t)^{\frac{1}{\delta_1}} + S_2(t)^{\frac{1}{1+\delta_2}} N_2(t - 1/2, t)^{\frac{1}{1+\delta_2}} \right) \left(\|\bar{q}\|_{L_\phi^2(U \times (t-\frac{1}{2}, t))} + \|\bar{q}\|_{L_\phi^2(U \times (t-\frac{1}{2}, t))}^{\frac{\delta_2}{1+\delta_2}} \right) \\ &\leq C N_2(t - 1/2, t)^{\frac{1}{1+\delta_2}} \left(S_2(t)^{\frac{1}{\delta_1}} + S_2(t)^{\frac{1}{1+\delta_2}} \right) \left(1 + \|\bar{q}\|_{L_\phi^2(U \times (t-\frac{1}{2}, t))} \right), \end{aligned}$$

where

$$S_2(t) = \left(B_* + \sup_{\tau \in [t-1/4, t]} \int_U W_1(x) |\nabla p(x, \tau)|^{2-a} dx d\tau \right)^{\frac{ar'}{4(2-a)}}.$$

Since $1 + \delta_2 > \delta_1$ and $S_2(t) \geq 1$, it follows that

$$\|\bar{q}\|_{L^\infty(U \times (t-\frac{1}{4}, t))} \leq CN_2(t-1/2, t)^{\frac{1}{1+\delta_2}} S_2(t)^{\frac{1}{\delta_1}} \left(1 + \|\bar{q}\|_{L_\phi^2(U \times (t-\frac{1}{2}, t))}\right). \quad (5.35)$$

Using (3.8) to estimate $S_2(t)$, and using (3.6), (3.2) to estimate $\|\bar{q}\|_{L_\phi^2(U \times (t-\frac{1}{2}, t))}$ in (5.35) yield

$$\begin{aligned} \|\bar{q}\|_{L^\infty(U \times (t-\frac{1}{4}, t))} &\leq CN_2(t-1/2, t)^{\frac{1}{1+\delta_2}} \\ &\cdot \left(\|\bar{p}(0)\|_{L_\phi^2(U)}^2 + \mathcal{M}(t-1)^{\frac{2}{2-a}} + \sup_{\tau \in [t-1/4, t]} \mathcal{M}(\tau)^{\frac{2}{2-a}} + \int_{t-\frac{5}{4}}^t (G(\tau) + G_1(\tau)) d\tau \right)^{\kappa_5 + \frac{1}{2}}. \end{aligned}$$

Since $\mathcal{M}(t) \geq M(\tau) \geq G(\tau) \geq 1$ for all $\tau \in [t - \frac{5}{4}, t]$, we obtain (5.31) consequently.

(ii) Taking limit superior of (5.35), we have

$$\limsup_{t \rightarrow \infty} \|\bar{q}\|_{L^\infty(U \times (t-\frac{1}{4}, t))} \leq C \limsup_{t \rightarrow \infty} N_2(t-1/2, t)^{\frac{1}{1+\delta_2}} \limsup_{t \rightarrow \infty} S_2(t)^{\frac{1}{\delta_1}} \left(1 + \limsup_{t \rightarrow \infty} \|\bar{q}\|_{L_\phi^2(U \times (t-\frac{1}{2}, t))}\right). \quad (5.36)$$

Using (3.9) to estimate the limit superior of $S_2(t)$, and using (3.6), (3.3) to estimate the limit superior of $\|\bar{q}\|_{L_\phi^2(U \times (t-\frac{1}{2}, t))}$, we obtain

$$\begin{aligned} \limsup_{t \rightarrow \infty} \|\bar{q}\|_{L^\infty(U \times (t-\frac{1}{4}, t))} &\leq C \limsup_{t \rightarrow \infty} N_2(t-1/2, t)^{\frac{1}{1+\delta_2}} \left(\mathcal{A}^{\frac{2}{2-a}} + \limsup_{t \rightarrow \infty} \int_{t-1}^t G_1(\tau) d\tau \right)^{\kappa_5} \\ &\cdot \left(\mathcal{A}^{\frac{2}{2-a}} + \limsup_{t \rightarrow \infty} \int_{t-1}^t (G(\tau) + G_1(\tau)) d\tau \right)^{\frac{1}{2}}. \end{aligned} \quad (5.37)$$

Note that

$$\limsup_{t \rightarrow \infty} \int_{t-1}^t G(\tau) d\tau \leq \limsup_{t \rightarrow \infty} G(t) = \mathcal{A} \leq \mathcal{A}^{\frac{2}{2-a}}.$$

Then (5.37) implies

$$\limsup_{t \rightarrow \infty} \|\bar{q}\|_{L^\infty(U \times (t-\frac{1}{4}, t))} \leq C \limsup_{t \rightarrow \infty} N_2(t-1/2, t)^{\frac{1}{1+\delta_2}} \left(\mathcal{A}^{\frac{2}{2-a}} + \limsup_{t \rightarrow \infty} \int_{t-1}^t G_1(\tau) d\tau \right)^{\kappa_5 + \frac{1}{2}},$$

and (5.32) follows.

(iii) Using (3.10) to estimate $S_2(t)$, and combining (3.6) with (3.4) to estimate $\|\bar{q}\|_{L_\phi^2(U \times (t-\frac{1}{2}, t))}$ in (5.35), we have for sufficient large t that

$$\begin{aligned} \|\bar{q}\|_{L^\infty(U \times (t-\frac{1}{4}, t))} &\leq CN_2(t-1/2, t)^{\frac{1}{1+\delta_2}} \sup_{s \in [t-1/4, t]} \left(B_* + \mathcal{B}^{\frac{1}{1-a}} + G(s)^{\frac{2}{2-a}} + \int_{s-1}^s G_1(\tau) d\tau \right)^{\kappa_5} \\ &\cdot \left(1 + \mathcal{B}^{\frac{1}{1-a}} + G(t-1)^{\frac{2}{2-a}} + \int_{t-1}^t (G(\tau) + G_1(\tau)) d\tau \right)^{\frac{1}{2}}. \end{aligned}$$

Since $G(t) \geq B_* \geq 1$, we have

$$\|\bar{q}(t)\|_{L^\infty(U \times (t-\frac{1}{4}, t))} \leq CN_2(t-1/2, t)^{\frac{1}{1+\delta_2}} \left(\mathcal{B}^{\frac{1}{1-a}} + \sup_{\tau \in [t-\frac{5}{4}, t]} G(\tau)^{\frac{2}{2-a}} + \int_{t-\frac{5}{4}}^t G_1(\tau) d\tau \right)^{\kappa_5 + \frac{1}{2}}. \quad (5.38)$$

We apply Lemma 2.6 to estimate $G(\tau)$ for $\tau \in [t - \frac{5}{4}, t]$ in terms of \mathcal{B} as

$$G(\tau) \leq G(t) + \frac{5}{4}(\mathcal{B} + 1).$$

Similar to the proof of Theorem 4.2, we then obtain (5.33) from (5.38). \square

Acknowledgement. The authors would like to thank Phuc Nguyen for helpful discussions. LH acknowledges the support by NSF grant DMS-1412796.

References

- [1] AULISA, E., BLOSHANSKAYA, L., HOANG, L., AND IBRAGIMOV, A. Analysis of generalized Forchheimer flows of compressible fluids in porous media. *J. Math. Phys.* 50, 10 (2009), 103102:44pp.
- [2] BEAR, J. *Dynamics of Fluids in Porous Media*. American Elsevier Pub. Co., New York, 1972.
- [3] CELIK, E., AND HOANG, L. Generalized Forchheimer flows in heterogeneous porous media. submitted, preprint <http://arxiv.org/abs/1509.01317>.
- [4] CELIK, E., HOANG, L., AND KIEU, T. Generalized Forchheimer flows of isentropic gases. submitted, preprint <http://arxiv.org/abs/1504.00742>.
- [5] CIANCHI, A., EDMUNDS, D. E., AND GURKA, P. On weighted Poincaré inequalities. *Math. Nachr.* 180 (1996), 15–41.
- [6] DiBENEDETTO, E. *Degenerate parabolic equations*. Universitext. Springer-Verlag, New York, 1993.
- [7] DUC, D. M., PHUC, N. C., AND NGUYEN, T. V. Weighted Sobolev’s inequalities for bounded domains and singular elliptic equations. *Indiana Univ. Math. J.* 56, 2 (2007), 615–642.
- [8] FORCHHEIMER, P. Wasserbewegung durch Boden. *Zeit. Ver. Deut. Ing.* 45 (1901), 1781–1788.
- [9] FORCHHEIMER, P. *Hydraulik*. No. Leipzig, Berlin, B. G. Teubner. 1930. 3rd edition.
- [10] HOANG, L., AND IBRAGIMOV, A. Structural stability of generalized Forchheimer equations for compressible fluids in porous media. *Nonlinearity* 24, 1 (2011), 1–41.
- [11] HOANG, L., AND IBRAGIMOV, A. Qualitative study of generalized Forchheimer flows with the flux boundary condition. *Adv. Diff. Eq.* 17, 5–6 (2012), 511–556.
- [12] HOANG, L., IBRAGIMOV, A., KIEU, T., AND SOBOL, Z. Stability of solutions to generalized Forchheimer equations of any degree. *J. Math. Sci.* 210, 4 (2015), 476–544.
- [13] HOANG, L., AND KIEU, T. Interior estimates for generalized Forchheimer flows of slightly compressible fluids. submitted, preprint <http://arxiv.org/abs/1404.6517>.
- [14] HOANG, L., AND KIEU, T. Global estimates for generalized Forchheimer flows of slightly compressible fluids. *Journal d’Analyse Mathématique* (2015). accepted.
- [15] HOANG, L. T., KIEU, T. T., AND PHAN, T. V. Properties of generalized Forchheimer flows in porous media. *J. Math. Sci.* 202, 2 (2014), 259–332.
- [16] LADYŽENSKAJA, O. A., SOLONNIKOV, V. A., AND URAL’CEVA, N. N. *Linear and quasi-linear equations of parabolic type*. Translated from the Russian by S. Smith. Translations of Mathematical Monographs, Vol. 23. American Mathematical Society, Providence, R.I., 1968.

- [17] MUSKAT, M. *The flow of homogeneous fluids through porous media*. McGraw-Hill Book Company, inc., 1937.
- [18] NIELD, D. A., AND BEJAN, A. *Convection in porous media*, fourth ed. Springer-Verlag, New York, 2013.
- [19] PAYNE, L. E., SONG, J. C., AND STRAUGHAN, B. Continuous dependence and convergence results for Brinkman and Forchheimer models with variable viscosity. *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.* 455, 1986 (1999), 2173–2190.
- [20] SAWYER, E., AND WHEEDEN, R. L. Weighted inequalities for fractional integrals on Euclidean and homogeneous spaces. *Amer. J. Math.* 114, 4 (1992), 813–874.
- [21] STRAUGHAN, B. *Stability and wave motion in porous media*, vol. 165 of *Applied Mathematical Sciences*. Springer, New York, 2008.
- [22] STRAUGHAN, B. Structure of the dependence of Darcy and Forchheimer coefficients on porosity. *Internat. J. Engrg. Sci.* 48, 11 (2010), 1610–1621.
- [23] WARD, J. C. Turbulent flow in porous media. *Journal of the Hydraulics Division, Proc. Am. Soc. Civ. Eng.* 90(HY5) (1964), 1–12.